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A STOCHASTIC FINITE ELEMENT METHOD FOR STOCHASTIC PARABOLIC EQUATIONS DRIVEN BY PURELY SPATIAL NOISE

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Abstract. We consider parabolic SPDEs driven by purely spatial noise, and show the existence of solutions with random initial data and forcing terms. We perform error analysis for the semi-discrete stochastic finite element method applied to a class of equations with self-adjoint differential operators that are independent of time. The analysis employs the formal stochastic adjoint problem and the corresponding elliptic error estimates to obtain the optimal order of convergence (in space).

1. Introduction

In this paper, we discuss stochastic finite element approximations of the following parabolic SPDE driven by multiplicative purely spatial noise $\dot{W}(x)$,

$$\frac{du}{dt} + \mathcal{A}u + (\mathcal{M}u + g) \diamond \dot{W}(x) = f \quad \text{on } D \times (0, T] \tag{1.1}$$
$$u|_{\partial D} = 0$$
$$u|_{t=0} = v$$

where $\mathcal{A}, \mathcal{M}$ are second order partial differential operators, and $\diamond$ denotes the Wick product (see e.g. [4], [5], [6], [7]). The stochastic finite element method (SFEM) for elliptic equations has been studied in [14], where the error estimates were derived in an appropriately weighted stochastic space. The approach taken there was based on Malliavin calculus and the Wiener Chaos expansion (see e.g. [11], [12]) and that is also the approach we will adopt. In fact, to obtain error estimates in the parabolic case, we will make integral use of the results from the elliptic error estimates, both directly from [14] as well as further results which we will derive in this paper. Thus, our error analysis can be viewed as a stochastic generalization of the standard techniques from deterministic FEM theory for parabolic equations [13].

The Malliavin calculus provides a tool to investigate the SFEM in an analogous way to the deterministic FEM. This approach reexpresses the Wick product in

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the form of Malliavin divergence operator, $(\mathcal{M}u + g) \diamond \dot{W}(x) = \boldsymbol{\delta}_{\dot{W}(x)}(\mathcal{M}u + g)$. As shown in [10], (1.1) is equivalent to a lower triangular system of deterministic parabolic PDE, known as the *propagator system*. Thus, the SFEM discretizes the randomness by a Galerkin approximation of the propagator system, and thanks to the lower triangular property, the SFEM reduces to an iterative procedure of applying the deterministic FEM to each equation in the truncated propagator system recursively. As in the elliptic case, our parabolic error estimates are comprised of two terms. One term represents the error from the stochastic truncation, while the other term represents the error from the application of the deterministic FEM to each equation in the truncated propagator system. Our error estimates achieve optimal spatial order of convergence, by analogy with the deterministic case; that is, for the spatial variable of the error measured in the L^2 norm, the convergence is $\mathcal{O}(h^{m+1})$ for a solution u with spatial smoothness of H^{m+2} and with H^m-smooth time derivative u_t.

Since the spatial regularity of the solution is imperative for the fast convergence of finite element schemes, it is also necessary to determine when the weak solution of (1.1) is also smooth. We will see that certain compatibility conditions at time $t = 0$, beyond those required in the deterministic case, are necessary for higher regularity to hold. Existence and uniqueness results for (1.1) have been studied in [10] under the assumption that v and f are deterministic and $g \equiv 0$. The SFEM elliptic error estimates in [14] also considered deterministic forcing term. By the nature of our error analysis technique, the error estimates for the case of random input data may be obtained with equal ease as for deterministic input data. Thus, we immediately consider the error estimates for (1.1) with random input data, and for this to make sense, we extend the existence and uniqueness result to allow for v, f, g to be random.

The framework of the Malliavin calculus is briefly described in Section 2. Section 3 deals with the existence of solution of equation (1.1), and gives a discussion on when the solution will be smoother in the spatial variable. The stochastic finite element method is detailed in Section 4, in which the statement of the main theorem on the parabolic error estimate is given. Section 5 discusses two issues relating to the corresponding stochastic elliptic problem – the formal stochastic adjoint problem and the extensions of the SFEM error estimates for the stochastic elliptic problem, both of which are ingredients of the proof of the main theorem in Section 6.

2. The Malliavin Calculus Framework

In this section, we describe the Malliavin calculus framework that we will use in the rest of the paper. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\mathcal{F}$ is the σ-algebra generated by $\xi = \{\xi_k\}_{k\geq 1}$. Let $\mathcal{U}$ be a real separable Hilbert space with complete orthonormal basis $\{\mathfrak{u}_k\}_{k\geq 1}$. In particular, since we are considering purely spatial noise, we will take $\mathcal{U} = L^2(D)$, for a domain $D \subset \mathbb{R}^d$, and assume $\mathfrak{u}_k$ are smooth. The Gaussian white noise on $\mathcal{U}$ is $\dot{W}(x) := \sum_{k\geq 1} \xi_k \mathfrak{u}_k(x)$.

Given a real separable Hilbert space X, let $L_2(\Omega; X)$ be the Hilbert space of square-integrable $\mathcal{F}$-measurable X-valued random elements. The Cameron-Martin basis is $\Xi = \{\xi_\alpha, \alpha \in \mathcal{J}\}$, where $\xi_\alpha = \prod_{k\geq 1} \frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}}$, and H_n is the n-th Hermite polynomial, and $\mathcal{J} = \{\alpha = (\alpha_1, \alpha_2, \dots)\}$ is the set of multi-indices.

We now introduce the weighted Wiener Chaos spaces. Let $\mathcal{R}$ be a bounded linear operator on $L_2(\Omega)$ defined by $\mathcal{R}\xi_\alpha = r_\alpha \xi_\alpha$ for every $\alpha \in \mathcal{J}$, where the weights $\{r_\alpha, \alpha \in \mathcal{J}\}$ are positive numbers. The inverse operator $\mathcal{R}^{-1}$ is defined by $\mathcal{R}^{-1}\xi_\alpha = r_\alpha^{-1}\xi_\alpha$. We define the space $\mathcal{R}L_2(\Omega; X)$ as the closure of $L^2(\Omega)$ under the norm

$$\|f\|^2_{\mathcal{R}L_2(\Omega;X)} := \sum_{\alpha\in\mathcal{J}} \|f_\alpha\|^2_X r_\alpha^2$$

for $f = \sum_{\alpha\in\mathcal{J}} f_\alpha \xi_\alpha$. In other words, the elements of $\mathcal{R}L_2(\Omega; X)$ is identified with a formal series $\sum_{\alpha\in\mathcal{J}} f_\alpha\xi_\alpha$, where $\|f\|^2_{\mathcal{R}L_2(\Omega;X)} < \infty$. Clearly, $\mathcal{R}L_2(\Omega; X)$ is a Hilbert space with respect to $\|\cdot\|_{\mathcal{R}L_2(\Omega;X)}$. Suppose $X \hookrightarrow Y \hookrightarrow X'$ is a normal triple of Hilbert spaces. We define the space $\mathcal{R}^{-1}L_2(\Omega; X)$ as the dual of $\mathcal{R}L_2(\Omega; X')$ relative to the inner product in the space $L_2(\Omega; Y)$. The duality pairing is given by

$$\langle\langle f, g\rangle\rangle_{\mathcal{R}L_2(\Omega;X'),\mathcal{R}^{-1}L_2(\Omega;X)} := \mathbb{E}[\langle \mathcal{R}f_\alpha, \mathcal{R}^{-1}g_\alpha\rangle_{X',X}] = \sum_{\alpha\in\mathcal{J}} \langle f_\alpha, g_\alpha\rangle_{X',X}$$

for $f \in \mathcal{R}L_2(\Omega; X')$ and $g \in \mathcal{R}^{-1}L_2(\Omega; X)$. Similarly, $\mathcal{R}^{-1}L_2(\Omega; X')$ is defined as the dual of $\mathcal{R}L_2(\Omega; X)$ relative to the inner product in $L_2(\Omega; Y)$.

In our paper, we will consider only admissible weights of the form

$$r_\alpha^2 = \frac{\rho^\alpha}{|\alpha|!},$$

where $\rho = (\rho_1, \rho_2, \dots)$, and $\rho^\alpha := \prod_k \rho_k^{\alpha_k}$. This class of weights are natural for the multiplicative noise structure appearing with the second order operator $\mathcal{M}$ in (1.1) [10].

Next, we define the Malliavin derivative $\boldsymbol{D}_{\xi_k}$ and Malliavin divergence operator $\boldsymbol{\delta}_{\xi_k}$ as follows

$$\boldsymbol{D}_{\xi_k}(\xi_\alpha) := \sqrt{\alpha_k}\xi_{\alpha-\varepsilon_k}, \qquad \boldsymbol{\delta}_{\xi_k}(\xi_\alpha) := \sqrt{\alpha_k+1}\xi_{\alpha+\varepsilon_k}.$$

Here, ε_k is the multiindex with 1 in the k-th entry and zero elsewhere. The Malliavin derivative and Malliavin divergence operator can be extended to random elements in $\mathcal{R}L_2(\Omega; X)$. In particular, for $f \in \mathcal{R}L_2(\Omega; X\otimes\mathcal{U})$, $\boldsymbol{\delta}_{\dot W}(f)$ is the unique element of $\mathcal{R}L_2(\Omega; X)$ with the property

$$\langle\langle \boldsymbol{\delta}_{\dot W}(f), \varphi\rangle\rangle_{\mathcal{R}L_2(\Omega;X),\mathcal{R}^{-1}L_2(\Omega;X')} = \langle\langle f, \boldsymbol{D}_{\dot W}\varphi\rangle\rangle_{\mathcal{R}L_2(\Omega;X\otimes\mathcal{U}),\mathcal{R}^{-1}L_2(\Omega;X'\otimes\mathcal{U})}$$

for every $\varphi \in \mathcal{R}^{-1}L_2(\Omega; X')$ such that $\boldsymbol{D}_{\dot W}\varphi \in \mathcal{R}^{-1}L_2(\Omega; X' \otimes \mathcal{U})$. Thus, the Malliavin derivative and Malliavin divergence operator are adjoint to each other. For a given $g \in \mathcal{R}L_2(\Omega; X)$, we also write $\boldsymbol{\delta}_{\dot W}(g)$ to mean $\boldsymbol{\delta}_{\dot W}\big(\sum_\alpha \sum_k \sqrt{\alpha_k} g_\alpha \otimes \mathfrak{u}_k\xi_\alpha\big)$, and write $g_{k,\alpha} = \mathfrak{u}_k \otimes g_\alpha$.

3. The Stochastic Parabolic Problem.

In this section, we consider the stochastic parabolic problem with zero Dirichlet boundary conditions, and state the conditions needed for the existence of a weak solution as well as for a solution with higher spatial regularity.

Let $D \subset \mathbb{R}^d$ be a bounded domain, and let $\mathcal{A}$ be a second order elliptic operator from $H_0^1(D)$ into $H^{-1}(D)$, and $\mathcal{M}_k$, $k = 1, 2, \dots$, be bounded operators from $H_0^1(D)$ into $H^{-1}(D)$. We will assume that the boundary ∂D and the coefficients of $\mathcal{A}, \mathcal{M}_k$ are sufficiently smooth, and also that $\mathcal{A}, \mathcal{M}_k$ do not depend on time. In the future, we will encounter the constants C_A and $\lambda_k^{(r)}$, which arise in

$$\|w\|_{L^2(0,T;H_0^1(D))} \leq C_A(\|w_0\|_{L^2(D)} + \|f\|_{L^2(0,T;H^{-1}(D))})$$

for the weak solution w of the Dirichlet problem $\frac{dw}{dt} + \mathcal{A}w = f$ with $w(0) = w_0$; and in

$$\|\mathcal{M}_k w\|_{H^{r-2}(D)} \leq \lambda_k^{(r)} \|w\|_{H^r(D)}, \qquad \forall w \in H^r(D).$$

For brevity, we write $\lambda_k = \lambda_k^{(1)}$.

The stochastic parabolic problem is

$$\begin{aligned} \frac{du}{dt} + \mathcal{A}u + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}u + g) &= f \quad \text{on } D \times (0, T] \\ u|_{\partial D} &= 0 \\ u|_{t=0} &= v \end{aligned} \tag{3.1}$$

where $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \dots)$, and $\mathcal{M}u := \sum_k \mathcal{M}_k u \otimes \mathfrak{u}_k$. The input data (i.e. the initial conditions and forcing terms) are allowed to be random.

In the future, we will use shorthand to denote the spaces: for example, we will write $\mathcal{R}_\Omega L_T^2 H_X^{-1}$ to denote $\mathcal{R}L_2(\Omega; L^2((0,T); H^{-1}(D)))$. Also, H_{0X}^1 denotes $H_0^1(D)$.

Definition 3.1. A *weak solution* of (3.1), with $f, g \in \mathcal{R}_\Omega L_T^2 H_X^{-1}$ and $v \in \mathcal{R}_\Omega L_X^2$, is a process $u \in \mathcal{R}_\Omega L_T^2 H_{0X}^1$ such that for every $\phi \in \mathcal{R}_\Omega^{-1}$ with $\boldsymbol{D}_{\dot{W}}\phi \in \mathcal{R}_\Omega^{-1}\mathcal{U}$,

$$\langle\langle u(t), \phi\rangle\rangle = \langle\langle v, \phi\rangle\rangle - \int_0^t \langle\langle \mathcal{A}u + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}u + g),\, \phi\rangle\rangle ds + \int_0^t \langle\langle f, \phi\rangle\rangle ds \tag{3.2}$$

with equality in $L_T^2 H_{0X}^1$.

The Equivalence Theorem 3.2 relates the weak solution to the propagator system (3.3).

Theorem 3.2. *The process* $u = \sum_\alpha u_\alpha \xi_\alpha \in \mathcal{R}_\Omega L_T^2 H_{0X}^1$ *is a solution of* (3.1), *if and only if, for each* $\alpha \in \mathcal{J}$,

$$u_\alpha(t) = v_\alpha - \int_0^t \mathcal{A}u_\alpha(s) + \sum_{k \geq 1} \sqrt{\alpha_k}(\mathcal{M}_k u_{\alpha - \epsilon_k} + g_{k,\alpha-\epsilon_k})\, ds + \int_0^t f_\alpha(s) ds \tag{3.3}$$

holds in H_X^{-1} *for a.e.* $t \in [0, T]$.

Proof. See [10]. □

3.1. The existence and uniqueness theorem. The existence and uniqueness of a weak solution of (3.1) for v, f deterministic and $g \equiv 0$ has been shown in [10]. We show the existence theorem for when v, f, g may be random, and determine the conditions for the weighted spaces that u may belong to, in terms of the spaces that the input data belong to.

Theorem 3.3. *Let the weights $\mathcal{R}$, with $r_\alpha^2 = \frac{q^\alpha}{|\alpha|!}$, satisfy*

$$\sum_{k\geq 1} q_k C_A^2 \lambda_k^2 < 1. \tag{3.4}$$

(1) *If the data $v \in L_X^2$ and $f, g \in L_T^2 H_X^{-1}$ are deterministic, then there exists a unique weak solution $u \in \mathcal{R}_\Omega L_T^2 H_{0X}^1$, and*

$$\|u\|_{\mathcal{R}_\Omega L_T^2 H_{0X}^1} \leq C \left(\|v\|_{L_X^2} + \|f\|_{L_T^2 H_X^{-1}} + \|g\|_{L_T^2 H_X^{-1}} \right)$$

where C depends only on $\mathcal{R}, \mathcal{A}, \mathcal{M}$ and T.

(2) *Assume $v \in \bar{\mathcal{R}}_\Omega L_X^2$ and $f, g \in \bar{\mathcal{R}}_\Omega L_T^2 H_X^{-1}$ for some $\bar{r}_\alpha^2 = \frac{\rho^\alpha}{|\alpha|!}$. Also assume, in addition to (3.4), q_k are chosen to satisfy*

$$\sum_{k\geq 1} \frac{q_k}{\rho_k} < 1 \tag{3.5}$$

Then there exists a unique solution $u \in \mathcal{R}_\Omega L_T^2 H_{0X}^1$, and

$$\|u\|_{\mathcal{R}_\Omega L_T^2 H_{0X}^1} \leq C \left(\|v\|_{\bar{\mathcal{R}}_\Omega L_X^2} + \|f\|_{\bar{\mathcal{R}}_\Omega L_T^2 H_X^{-1}} + \|g\|_{\bar{\mathcal{R}}_\Omega L_T^2 H_X^{-1}} \right)$$

where C depends only on $\mathcal{R}, \bar{\mathcal{R}}, \mathcal{A}, \mathcal{M}$ and T.

Proof. The proof proceeds along the usual steps, (see e.g. [8] (Theorem 9.4) or [9] (Proposition 4.2)).

Step 1.

Assume v, f, g are non-random. This case has been studied in [10] for $g = 0$. The proof here is essentially the same. The propagator system is

$$\begin{aligned} u_{(0)}(t) &= v + \int_0^t \mathcal{A} u_{(0)}(s) + f(s) ds \\ u_{\epsilon_k}(t) &= \int_0^t \mathcal{A} u_{\epsilon_k}(s) + \big(\mathcal{M}_k u_{(0)}(s) + g_k(s)\big) ds \\ u_\alpha(t) &= \int_0^t \mathcal{A} u_\alpha(s) + \sum_k \sqrt{\alpha_k} \mathcal{M}_k u_{\alpha-\epsilon_k}(s) ds, \quad |\alpha| \geq 2 \end{aligned}$$

An explicit formula for the chaos coefficients is (c.f. proof of Theorem 3.11 in [10])

$$u_{(0)}(t) = \Phi_t v + \int_0^t \Phi_{t-s} f(s) ds$$

and for $|\alpha| = n$,

$$u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_0^t \int_0^{s_n} \dots \int_0^{s_2} \Phi_{t-s_n} \mathcal{M}_{k_{\sigma(n)}} \dots \Phi_{s_2-s_1} \big(\mathcal{M}_{k_{\sigma(1)}} u_{(0)}(s_1) + g_{k_{\sigma(1)}}\big) ds_1 \dots ds_n$$

Here, $K_\alpha = (k_1, \dots, k_{|\alpha|})$ is the characteristic set of α, $\mathcal{P}_n$ is the group of permutations of $\{1, \dots, n\}$, and Φ_t is the semigroup generated by $\mathcal{A}$.

By induction, and by application of the deterministic parabolic estimates, we obtain

$$\|u_{(0)}\|_{L^2_T H^1_{0X}} \leq C_A(\|v\|_{L^2_X} + \|f\|_{L^2_T H^{-1}_X})$$
$$\|u_\alpha\|_{L^2_T H^1_{0X}} \leq C_A M \frac{(\vec{\lambda} C_A)^\alpha |\alpha|!}{\sqrt{\alpha!}}(\|v\|_{L^2_X} + \|f\|_{L^2_T H^{-1}_X} + \|g\|_{L^2_T H^{-1}_X})$$

where $M = \sup_k(1 \vee \frac{\mu_k}{\lambda_k C_A})$, and μ_k is the constant in $\|g_k\|_{H^{-1}_X} \leq \mu_k \|g\|_{H^{-1}_X}$. So taking the weights to satisfy (3.4), it follows from Lemma B.1 that

$$\|u\|_{\mathcal{R}_\Omega L^2_T H^1_{0X}} \leq C(\|v\|_{L^2_X} + \|f\|_{L^2_T H^{-1}_X} + \|g\|_{L^2_T H^{-1}_X})$$

C depends only on $\mathcal{R}, \mathcal{A}, \mathcal{M}$ and T.

Step 2.

Fix an arbitrary $\alpha^* \in \mathcal{J}$. Assume $v = V\xi_{\alpha^*}, f = F\xi_{\alpha^*}, g = G\xi_{\alpha^*}$; in other words, the randomness of the data is localized to a single mode. Let $u[\alpha^*; V, F, G](t, x)$ be the solution. By linearity, the chaos expansion coefficients with indices of the form $\alpha^* + \alpha$ satisfy

$$\frac{u_{\alpha^*+\alpha}[\alpha^*; V, F, G]}{\sqrt{(\alpha^* + \alpha)!}} = \frac{u_\alpha[(0); \frac{V}{\sqrt{\alpha^*!}}, \frac{F}{\sqrt{\alpha^*!}}, \frac{G}{\sqrt{\alpha^*!}}]}{\sqrt{\alpha!}}$$

and are zero otherwise. Then

$$\begin{aligned}
&\int_0^T \|u[\alpha^*; V, F, G](t)\|^2_{\mathcal{R}_\Omega H^1_{0X}} dt \\
&= \sum_\alpha \frac{q^{\alpha^*+\alpha}}{|\alpha^* + \alpha|!} \frac{(\alpha^* + \alpha)!}{\alpha!} \left\| u_\alpha \left[(0); \frac{V}{\sqrt{\alpha^*!}}, \frac{F}{\sqrt{\alpha^*!}}, \frac{G}{\sqrt{\alpha^*!}}\right] \right\|^2_{L^2_T H^1_{0X}} \\
&= \sum_\alpha \frac{q^{\alpha^*+\alpha}}{|\alpha|!|\alpha^*|!} \frac{|\alpha|!|\alpha^*|!}{|\alpha^* + \alpha|!} \frac{(\alpha^* + \alpha)!}{\alpha!\alpha^*!} \|u_\alpha[(0); V, F, G]\|^2_{L^2_T H^1_{0X}} \\
&\leq \frac{q^{\alpha^*}}{|\alpha^*|!} \|u[(0); V, F, G]\|^2_{\mathcal{R}_\Omega L^2_T H^1_{0X}}
\end{aligned}$$

where the last inequality follows by Lemma B.2.

Step 3.

For the general case with random data, assume $v \in \bar{\mathcal{R}}_\Omega L^2_X$ and $f, g \in \bar{\mathcal{R}}_\Omega L^2_T H^{-1}_X$. The solution can be written as

$$u = \sum_{\alpha^*} u[\alpha^*; v_{\alpha^*}, f_{\alpha^*}, g_{\alpha^*}]$$

Using the estimates from Step 2,

$$\|u\|_{\mathcal{R}_\Omega L^2_T H^1_{0X}} \leq \sum_{\alpha^*} \|u[\alpha^*; v_{\alpha^*}, f_{\alpha^*}, g_{\alpha^*}]\|_{\mathcal{R}_\Omega L^2_T H^1_{0X}}$$

$$
\begin{aligned}
&\leq C\left(\sum_{\alpha^*}\frac{q^{\alpha^*}}{|\alpha^*|!}\frac{|\alpha^*|!}{\rho^{\alpha^*}}\right)^{1/2} \\
&\quad\times\left(\sum_{\alpha^*}\frac{\rho^{\alpha^*}}{|\alpha^*|!}\left(\|v_{\alpha^*}\|_{L^2_X}+\|f_{\alpha^*}\|_{L^2_TH^{-1}_X}+\|g_{\alpha^*}\|_{L^2_TH^{-1}_X}\right)^2\right)^{1/2} \\
&\leq C\left(\|v\|_{\bar{\mathcal{R}}_\Omega L^2_X}+\|f\|_{\bar{\mathcal{R}}_\Omega L^2_TH^{-1}_X}+\|g\|_{\bar{\mathcal{R}}_\Omega L^2_TH^{-1}_X}\right)
\end{aligned}
$$

where we have applied Cauchy-Schwartz inequality in the second inequality. The convergence of $\left(\sum_{\alpha^*}\frac{q^{\alpha^*}}{|\alpha^*|!}\frac{|\alpha^*|!}{\rho^{\alpha^*}}\right)$ follows from a sufficient condition such as (3.5).

Clearly, $\mathcal{R}\supseteq\bar{\mathcal{R}}$, so u is a weak solution of (3.1) in the sense of Definition 3.1. Uniqueness follows from the uniqueness of each equation in the propagator system. □

Remark 3.4. (1) The validity of the assumption that $M:=\sup_k(1\vee\frac{\mu_k}{\lambda_k})<\infty$ arises in some common examples. For example, taking $\mathcal{M}_k\phi=\mathrm{u}_k\Delta\phi$ and $g_k=\mathrm{u}_kg$, we have that μ_k,λ_k are both $\sim\mathcal{O}(k)$. If $M=\infty$, then in the estimate for $\|u_\alpha\|_{L^2_TH^1_{0X}}$ in Step 1, we should replace the factor $M\vec{\lambda}^\alpha$ by $(\vec{\lambda}C_A\vee\vec{\mu})^\alpha$, and use the criterion $\sum_k q_k(\lambda_kC_A\vee\mu_k)^2<1$ in place of (3.4).

(2) If the input data is non-random, then it belongs to any weighted space $\bar{\mathcal{R}}$ for any ρ. In this case, condition (3.5) is automatically satisfied, and the condition for optimal solution weights $\mathcal{R}$ reduces to (3.4) alone.

3.2. Higher spatial regularity of solutions. The weak solution of (3.1) is a generalized process on H^1_{0X}. We can ask the question of when the solution is actually a generalized process on a better space H^m_X. This result is actually important for the error analysis of the stochastic finite element method later on, which requires that u,u_t,u_{tt} be L_2 functions in the spatial variable. This higher spatial regularity of the solution follows from analogous results in the deterministic case, but comes at the expense of worsening the weights $\mathcal{R}$.

We first recall a higher regularity result in the deterministic case, in which certain *compatibility conditions* are necessary conditions for higher spatial regularity.

Theorem 3.5. *(Evans [3], Thm 5 and 6 in §7.1.3). Suppose $u\in L^2_TH^1_{0X}$ with $u_t\in L^2_TH^{-1}_X$ is the weak solution of*

$$
\begin{cases}
u_t+\mathcal{A}u=f & \text{in } D\times(0,T]\\
u=0 & \text{on } \partial D\times[0,T]\\
u=v & \text{on } D\times\{t=0\}
\end{cases}
$$

(i) Assume

$$v\in H^1_{0X},\qquad f\in L^2_TL^2_X.$$

Then in fact $u\in L^2_TH^2_X\cap L^\infty_TH^1_{0X}$ and $u_t\in L^2_TL^2_X$, and

$$\operatorname*{ess\,sup}_{0\leq t\leq T}\|u(t)\|_{H^1_{0X}}+\|u\|_{L^2_TH^2_X}+\|u_t\|_{L^2_TL^2_X}\leq C_0^{reg}\left(\|v\|_{H^1_{0X}}+\|f\|_{L^2_TL^2_X}\right)$$

where the constant C_0^{reg} depends only on D,T and $\mathcal{A}$.

(ii) Fix $m \geq 1$. Assume

$$v \in H_X^{2m+1}, \qquad \frac{d^k f}{dt^k} \in L_T^2 H_X^{2m-2k} \quad \text{for } k = 0, \dots, m$$

and suppose the m-th order compatibility conditions hold:

$$\begin{cases} v_0 := V_0 \in H_{0X}^1, \quad V_1 := f(0) - \mathcal{A}V_0 \in H_{0X}^1, \\ \dots, V_m := \frac{d^{m-1}f}{dt^{m-1}} - \mathcal{A}V_{m-1} \in H_{0X}^1 \end{cases}$$

Then $\frac{d^k u}{dt^k} \in L_T^2 H_X^{2m+2-2k}$ for $k = 0, \dots m+1$, and

$$\sum_{k=0}^{m} \left\| \frac{d^k u}{dt^k} \right\|_{L_T^2 H_X^{2m+2-2k}} \leq C_m^{reg} \left(\|v\|_{H_X^{2m+1}} + \sum_{k=0}^{m} \left\| \frac{d^k f}{dt^k} \right\|_{L_T^2; H_X^{2m-2k}} \right)$$

where the constant C_m^{reg} depends only on m, D, T and $\mathcal{A}$.

From Theorem 3.5(i), we can obtain the following higher regularity result for the stochastic equation (3.1), with deterministic input data. The case of random data can be shown in the same way as Steps 2 and 3 in the proof of Theorem 3.3.

Below, we will encounter the constant $\theta_k^{(r)}$, $r = 0, 1, \dots$, arising in $\|g_k\|_{H_X^r} \leq \theta_k^{(r)} \|g\|_{H_X^r}$.

Corollary 3.6. *Suppose $u \in \mathcal{R}_\Omega L_T^2 H_{0X}^1$ is the weak solution of the SPDE* (3.1). *Also assume that v, f, g are deterministic with*

$$v \in H_{0X}^1, \quad \text{and} \quad f, g \in L_T^2 L_X^2$$

Then for the weights $\tilde{\mathcal{R}}$ satisfying

$$\sum_k \tilde{\rho}_k (\lambda_k^{(2)} C_0^{reg})^2 < 1,$$

the weak solution $u \in \tilde{\mathcal{R}}_\Omega L_T^2 H_X^2$ and

$$\|u\|_{\tilde{\mathcal{R}}_\Omega L_T^2 H_X^2} \leq C \left(\|v\|_{H_{0X}^1} + \|f\|_{L_T^2 L_X^2} + \|g\|_{L_T^2 L_X^2} \right)$$

Proof. The proof is similar to the proof of Theorem 3.3. The estimates for each u_α are obtained by applying Theorem 3.5(i) to the propagator system. □

No special compatibility conditions were necessary for Corollary 3.6, but it is unable to ensure boundedness of u_{tt}. Thus, we next show how to obtain a smoother solution and the boundedness of u_{tt} using the 1st order compatibility conditions.

Corollary 3.7. *Suppose $u \in \mathcal{R}_\Omega L_T^2 H_{0X}^1$ is the weak solution of the SPDE* (3.1). *Also assume that v, f, g are deterministic with*

$$v \in H_X^3, \quad \text{and} \quad f, g \in L_T^2 H_X^2, \quad \text{and} \quad \frac{df}{dt}, \frac{dg}{dt} \in L_T^2 L_X^2,$$

and that the 1st order compatibility conditions hold for $\{v, f, g_k\}$:

$$\begin{cases} v \in H_{0X}^1, \quad f(0) - \mathcal{A}v \in H_{0X}^1, \\ \mathcal{M}_k v + g_k(0) \in H_{0X}^1 \quad \forall k = 1, 2, \dots \end{cases} \tag{3.6}$$

Then for the weights $\mathcal{R}'$ satisfying

$$\sum_k \rho'_k \Big((\lambda_k^{(4)} \vee \lambda_k^{(2)}) C_1^{reg} \Big)^2 < 1, \tag{3.7}$$

the weak solution $u \in \mathcal{R}'_\Omega L_T^2 H_X^4$, $u_t \in \mathcal{R}'_\Omega L_T^2 H_X^2$ and $u_{tt} \in \mathcal{R}'_\Omega L_T^2 L_X^2$ and

$$\begin{aligned} &\|u\|_{\mathcal{R}'_\Omega L_T^2 H_X^4} + \|u_t\|_{\mathcal{R}'_\Omega L_T^2 H_X^2} + \|u_{tt}\|_{\mathcal{R}'_\Omega L_T^2 L_X^2} \\ &\quad \le C\big(\|v\|_{H_X^3} + \|f\|_{L_T^2 H_X^2} + \|g\|_{L_T^2 H_X^2} + \|f_t\|_{L_T^2 L_X^2} + \|g_t\|_{L_T^2 L_X^2} \big) \end{aligned}$$

Proof. For $\alpha = (0)$, the (deterministic) compatibility conditions hold, and from Theorem 3.5(ii),

$$\begin{aligned} &\|u_{(0)}\|_{L_T^2 H_X^4} + \|u_{(0),t}\|_{L_T^2 H_X^2} + \|u_{(0),tt}\|_{L_T^2 L_X^2} \\ &\quad \le C_1^{reg} \left(\|v\|_{H_X^3} + \|f\|_{L_T^2 H_X^2} + \|f_t\|_{L_T^2 L_X^2} \right). \end{aligned}$$

For $\alpha = \varepsilon_k$, since we have assumed the coefficients of $\mathcal{M}_k$ to be sufficiently smooth (e.g., at least $W_X^{3,\infty}$), so $u_{(0)} \in L_T^2 H_X^4$ implies that $\mathcal{M}_k u_{(0)} + g_k \in L_T^2 H_X^2$, and $u_{(0),t} \in L_T^2 H_X^2$ implies that $(\mathcal{M}_k u_{(0)} + g_k)_t \in L_T^2 L_X^2$. The compatibility conditions for $(\mathcal{M}_k u_{(0)} + g_k)\big|_{t=0} = \mathcal{M}_k v + g_k(0)$ are also satisfied. Again applying Theorem 3.5(ii),

$$\begin{aligned} &\|u_{\varepsilon_k}\|_{L_T^2 H_X^4} + \|(u_{\varepsilon_k})_t\|_{L_T^2 H_X^2} + \|(u_{\varepsilon_k})_{tt}\|_{L_T^2 L_X^2} \\ &\le C_1^{reg} \left(\lambda_k^{(4)} \|u_{(0)}\|_{L_T^2 H_X^4} + \theta_k^{(2)} \|g\|_{L_T^2 H_X^2} + \lambda_k^{(2)} \|(u_{(0)})_t\|_{L_T^2 H_X^2} + \theta_k^{(0)} \|g_t\|_{L_T^2 L_X^2} \right) \\ &\le C_1^{reg} (\lambda_k^{(4)} \vee \lambda_k^{(2)}) \tilde{\tilde{M}} \\ &\qquad \times \left(\|v\|_{H_X^3} + \|f\|_{L_T^2 H_X^2} + \|f_t\|_{L_T^2 L_X^2} + \|g\|_{L_T^2 H_X^2} + \|g_t\|_{L_T^2 L_X^2} \right) \end{aligned}$$

where $\tilde{\tilde{M}} = \sup_k \left\{ 1 \vee \frac{(\lambda_k^{(4)} \vee \lambda_k^{(2)})}{(\theta_k^{(2)} \vee \theta_k^{(0)}) C_1^{ref}} \right\}$. (The remark following Theorem 3.3 applies.)

For $|\alpha| \ge 2$, we have $\mathcal{M}_k u_{\alpha - \varepsilon_k} \in L_T^2 H_X^2$ and $(\mathcal{M}_k u_{\alpha - \varepsilon_k})_t \in L_T^2 L_X^2$. The compatibility conditions hold trivially, since $u_{\alpha - \varepsilon_k}\big|_{t=0} \equiv 0$ whenever $|\alpha| \ge 2$. The usual computations give the estimates,

$$\begin{aligned} &\|u_\alpha\|_{L_T^2 H_X^4} + \|u_{\alpha,t}\|_{L_T^2 H_X^2} + \|u_{\varepsilon_k,tt}\|_{L_T^2 L_X^2} \\ &\le C_1^{reg} \tilde{\tilde{M}} \frac{\big(C_1^{reg} (\vec{\lambda^{(4)}} \vee \vec{\lambda^{(2)}}) \big)^\alpha |\alpha|!}{\sqrt{\alpha!}} \\ &\qquad \times \left(\|v\|_{H_X^3} + \|f\|_{L_T^2 H_X^2} + \|f_t\|_{L_T^2 L_X^2} + \|g\|_{L_T^2 H_X^2} + \|g_t\|_{L_T^2 L_X^2} \right). \end{aligned}$$

The weighted norm $\|u\|_{\mathcal{R}'_\Omega L_T^2 H_X^4} < \infty$ provided (3.7) holds. □

Due to the lower triangular property of the propagator system, the first order compatibility conditions for the stochastic parabolic equation involve additional conditions on the input data compared to the deterministic case. If the input data is smoother than what is assumed in Corollary 3.7, additional compatibility conditions on the derivatives $\{D^\gamma v, D^\gamma f, D^\gamma g\}$ are required to further raise the spatial regularity of u, u_t and u_{tt}, even if the boundedness of time derivatives beyond u_{tt} are not needed. If the input data is random, similar arguments as Steps 2 and 3 in Theorem 3.3 extends Corollary 3.7 to the random input data case, this time with additional compatibility conditions on the modes $\{v_\alpha, f_\alpha, g_\alpha\}$. These results are summarized in the following theorem.

Theorem 3.8. *Suppose $u \in \mathcal{R}_\Omega L^2_T H^1_{0X}$ is the weak solution of the SPDE* (3.1). *For fixed $m \geq 2$, also assume that*

$$v \in \bar{\mathcal{R}}_\Omega H^{m+1}_X, \quad \text{and} \quad f, g \in \bar{\mathcal{R}}_\Omega L^2_T H^m_X, \quad \text{and} \quad \frac{df}{dt}, \frac{dg}{dt} \in \bar{\mathcal{R}}_\Omega L^2_T H^{m-2}_X,$$

and that the compatibility conditions (3.6) *hold for $\{D^\gamma v_\alpha, D^\gamma f_\alpha, D^\gamma g_{k,\alpha}\}$, for all $\alpha \in \mathcal{J}$, and all indices $\gamma = (\gamma_1, \ldots, \gamma_d)$ with $|\gamma| \leq m-2$.*

Then for the weights $\mathcal{R}'$ satisfying

$$\sum_k \rho'_k \Big((\lambda^{(4)}_k \vee \lambda^{(2)}_k) C^{reg}_1 \Big)^2 < 1 \quad \text{and} \quad \sum_k \frac{q'_k}{\rho_k} < 1, \tag{3.8}$$

the weak solution $u \in \mathcal{R}'_\Omega L^2_T H^{m+2}_X$, $u_t \in \mathcal{R}'_\Omega L^2_T H^m_X$ and $u_{tt} \in \mathcal{R}'_\Omega L^2_T H^{m-2}_X$ and

$$\begin{aligned} &\|u\|_{\mathcal{R}'_\Omega L^2_T H^{m+2}_X} + \|u_t\|_{\mathcal{R}'_\Omega L^2_T H^m_X} + \|u_{tt}\|_{\mathcal{R}'_\Omega L^2_T H^{m-2}_X} \\ &\leq C\big(\|v\|_{\bar{\mathcal{R}}_\Omega H^{m+1}_X} + \|f\|_{\bar{\mathcal{R}}_\Omega L^2_T H^m_X} + \|g\|_{\bar{\mathcal{R}}_\Omega L^2_T H^m_X} \\ &\quad + \|f_t\|_{\bar{\mathcal{R}}_\Omega L^2_T H^{m-2}_X} + \|g_t\|_{\bar{\mathcal{R}}_\Omega L^2_T H^{m-2}_X}\big). \end{aligned}$$

This is the basic structure of the smoothness assumption we will make when performing the error analysis for the SFEM.

4. Stochastic Finite Element Method

The stochastic finite element method adopts the same strategy as the deterministic situation, by casting the weak formulation of the problem into a finite dimensional setting. We consider only the semi-discrete case in this paper, where we have kept the time variable continuous and discretized the stochastic and spatial variables only, thus yielding a system of ODE; this discretization is achieved by Galerkin approximation in randomness and finite element approximation in space. Subsequently, the fully discrete case can be done by applying a suitable time stepping algorithm to the system of ODE.

Finite element approximation in space. We recall the usual finite element set up. Let $(K_{ref}, \mathcal{P}, \mathcal{N})$ be a reference finite element. Let $\mathcal{T}_h$ be a family of quasi-uniform triangulations. For $K \in \mathcal{T}_h$, let $S^K_h = \{z : z \circ F^{-1}_K \in \mathcal{P}(K_{ref})\}$ where $F_K : K_{ref} \to K$ is affine. The finite element space is

$$S_h = \{z \in H^1_0(D) : z|_K \in S^K_h, K \in \mathcal{T}_h\}$$

A property of S_h we assume is that there exists $r \geq 2$ such that for h small,

$$\inf_{z_h \in S_h} \big\{ \|v - z_h\|_{L_2} + h\|\nabla(v - z_h)\|_{L_2} \big\} \leq C h^s \|v\|_{H^s}, \quad \text{for } 1 \leq s \leq r \tag{4.1}$$

whenever $v \in H^r \cap H^1_0$ [13]. We also assume that, in particular, S_h consists of piecewise polynomials of degree at most $r-1$, so that the inverse inequality holds,

$$\|\nabla z_h\|_{L^2} \leq C h^{-1} \|z_h\|_{L^2}, \quad \forall z_h \in S_h.$$

We denote the FE basis of S_h by $\{\Phi_l\}_{l=1,\dots,\dim S_h}$.

Galerkin approximation in randomness. Letting

$$\mathcal{J}_{M,n} := \{\gamma \in \mathcal{J} \,:\, |\gamma| \le n,\, \dim(\gamma) \le M\},$$

we define the truncated Wiener chaos space

$$S^{M,n} = \Big\{ f = \sum_{\gamma \in \mathcal{J}_{M,n}} f_\gamma \xi_\gamma \,:\, f_\gamma \in \mathbb{R} \Big\}.$$

SFEM formulation. The stochastic finite element method is

Find $u_h^{M,n} \in S_h \otimes S^{M,n}$ such that

$$\langle\langle \frac{du_h^{M,n}}{dt}, z_h \rangle\rangle_{\mathcal{R}_\Omega^{\mp 1} L_X^2} + \langle\langle \mathcal{A} u_h^{M,n} + \sum_{k=1}^{M} \boldsymbol{\delta}_{\xi_k} (\mathcal{M}_k u_h^{M,n} + g_k), z_h \rangle\rangle_{\mathcal{R}_\Omega^{\mp 1} H_X^{\mp 1}}$$
$$= \langle\langle f, z_h \rangle\rangle_{\mathcal{R}_\Omega^{\mp 1} H_X^{\mp 1}} \tag{4.2}$$

for all $z_h \in S^{M,n} \otimes S_h$, and for every $t \in [0,T]$.

Denote $u_h^{M,n} = \sum_{\gamma \in \mathcal{J}_{M,n}} \hat{u}_\gamma \xi_\gamma$. Solving (3.2) via the SFEM is equivalent to solving each equation in the truncated propagator system via FEM: for $\alpha \in \mathcal{J}_{M,n}$,

$$\big(\frac{d\hat{u}_{(0)}}{dt}, z_h\big) + \boldsymbol{A}[\hat{u}_{(0)}, z_h] = \langle f_{(0)}, z_h \rangle, \tag{4.3}$$

$$\big(\frac{d\hat{u}_\alpha}{dt}, z_h\big) + \boldsymbol{A}[\hat{u}_\alpha, z_h] + \sum_{k=1}^{M} \sqrt{\alpha_k} \big(\boldsymbol{M}_k[\hat{u}_{\alpha-\varepsilon_k}, z_h] + \langle g_{k,\alpha-\varepsilon_k}, z_h \rangle\big) = \langle f_\alpha, z_h \rangle, \tag{4.4}$$

for all $z_h \in S_h$, with initial conditions $\hat{u}_\alpha|_{t=0} = v_{h,\alpha}^{M,n}$. The bilinear forms $\boldsymbol{A}, \boldsymbol{M}_k$ are are the bilinear forms associated with $\mathcal{A}, \mathcal{M}_k$.

The algorithm. Next, we write out the SFEM algorithm explicitly to show the resulting system of ODE. We define the mass and stiffness matrices identically to the usual FEM case, and also a noise matrix arising from the stochastic term:

$$\mathbb{M}_{l'l}^{mass} = (\Phi_l, \Phi_{l'}), \qquad \mathbb{M}_{l'l}^{stiff} = \boldsymbol{A}[\Phi_l, \Phi_{l'}], \qquad \mathbb{M}_{k;l'l}^{noise} = \boldsymbol{M}_k[\Phi_l, \Phi_{l'}].$$

The lower triangular discrete propagator system is solved iteratively. For the numerical solution $u_h^{M,n}(t) = \sum_{\gamma \in J_{M,n}} \sum_{l=1}^{\dim S_h} \hat{u}_{\gamma,l}(t) \Phi_l \xi_\gamma$, let the solution vector be $\vec{\hat{u}}_\gamma = (\hat{u}_{\gamma,1}, \dots, \hat{u}_{\gamma,\dim S_h})^T$. Then, for $\gamma = (0)$,

$$\mathbb{M}^{mass} (\vec{\hat{u}}_{(0)})_t + \mathbb{M}^{stiff} \vec{\hat{u}}_{(0)} = \vec{f}_{(0)}$$

and for $|\gamma| \ge 1$,

$$\mathbb{M}^{mass} (\vec{\hat{u}}_\gamma)_t + \mathbb{M}^{stiff} \vec{\hat{u}}_\gamma + \sum_k \sqrt{\gamma_k} \left(\mathbb{M}^{noise} \vec{\hat{u}}_{\gamma-\varepsilon_k} + \vec{g}_{k,\gamma-\varepsilon_k} \right) = \vec{f}_\gamma$$

where

$$\vec{f}_\gamma = (\langle f_\gamma, \Phi_1 \rangle, \dots, \langle f_\gamma, \Phi_{\dim S_h} \rangle)^T, \quad \text{and}$$
$$\vec{g}_{k,\gamma} = (\langle g_{k,\gamma}, \Phi_1 \rangle, \dots, \langle g_{k,\gamma}, \Phi_{\dim S_h} \rangle)^T.$$

Error analysis. The error analysis relies on decomposing the approximation error into two contributors

$$u_h^{M,n}(t) - u(t) = (u_h^{M,n} - U) + (U - u) = \theta(t) + \pi(t)$$

for some carefully chosen U in a subspace of $\mathcal{R}_\Omega L_T^2 H_{0X}^1$. Two possible choices of U are

(1) $U = \bar{u} := \sum_{\gamma \in \mathcal{J}_{M,n}} u_\gamma \xi_\gamma$, the truncated Wiener chaos expansion of u. Then θ_γ is the error from the FEM approximation of the γ-th equation in (4.3), and π is the error from truncating the Wiener Chaos expansion of u.

(2) $U = \Pi_h^{M,n} u$, where $\Pi_h^{M,n} : \mathcal{R}L_2(\Omega; H_0^1) \to S^{M,n} \otimes S_h$ is the SFEM solution operator for the corresponding stochastic elliptic problem. Then π is the error from the associated elliptic problem, whereas θ is the error between the parabolic and elliptic approximations.

We will adopt the second approach for our error analysis.

4.1. Error analysis. For the error analysis, we assume $\mathcal{A}$ and $\mathcal{M}$ take the form

$$\mathcal{A}u = -\sum_{i,j} D_i(a^{ij}(x) D_j u)$$
$$\mathcal{M}_k u = \sum_{i,j} D_i(\sigma_k^{ij}(x) D_j u)$$

where a^{ij}, σ_k^{ij} are measurable and bounded in $\bar{D}$, and $\mathcal{A}$ is uniformly elliptic with coercivity constant C_A^{coerc}. Also, $C_A^{ellip} = (C_A^{coerc})^{-1}$ is the constant in $\|w\|_{H_{0X}^1} \leq C_A^{ellip} \|f\|_{H_X^{-1}}$ for the solution of the zero Dirichlet problem $\mathcal{A}w = f$. We also assume for simplicity that $g \equiv 0$.

We now derive the error estimates for the parabolic equation (3.1). The error $e_h(t) := u_h^{M,n}(t) - u(t)$ will be measured in the $\mathcal{R}_\Omega L_X^2$-norm for every $t \in (0, T]$, and we will determine the conditions on the weights $\mathcal{R}$ that admit these error estimates.

Theorem 4.1. *Let $m \geq 2$ be an even integer. Assume for the input data*

$$v \in \bar{\mathcal{R}}_\Omega H_X^{m+1}, \quad f \in \bar{\mathcal{R}}_\Omega L_T^2 H_X^m, \quad f_t \in \bar{\mathcal{R}}_\Omega L_T^2 H_X^{m-2},$$

with weights $\bar{r}_\alpha^2 = \frac{\bar{\rho}^\alpha}{|\alpha|!}$, and assume that the appropriate compatibility conditions hold, so that

$$u \in \mathcal{R}'_\Omega L_T^2 H_{0X}^1 \cap \mathcal{R}'_\Omega L_T^2 H_X^{m+2}, \quad u_t \in \mathcal{R}'_\Omega L_T^2 H_X^{-1} \cap \mathcal{R}'_\Omega L_T^2 H_X^m,$$
$$u_{tt} \in \mathcal{R}'_\Omega L_T^2 H_X^{m-2},$$

where the weights ${\rho'_\alpha}^2 = \frac{\rho'^\alpha}{|\alpha|!}$ are chosen using the conditions (3.5) *and* (3.7)*. Also assume, for simplicity, that the discretized initial condition is $v_h = \Pi_h^{M,n} v$. Then, for every $t \in (0, T]$, we have the error estimate for the SFEM solution $u_h^{M,n}(t)$,*

$$\|e_h(t)\|_{\mathcal{R}_\Omega L_X^2} \leq C_{M,n} h^{m+1} \left(\|u_t\|_{\mathcal{R}_\Omega L_T^2 H_X^m} + \|u(t)\|_{\mathcal{R}_\Omega H_X^{m+1}} \right) \tag{4.5}$$
$$+ C Q_{M,n}(\mathcal{R}, \mathcal{R}') \left(\|f_t - u_{tt}\|_{\mathcal{R}'_\Omega L_T^2 H_X^{-1}} + \|f(t) - u_t(t)\|_{\mathcal{R}'_\Omega H_X^{-1}} \right)$$

where the weights $\mathcal{R}$, ${r_\alpha}^2 = \frac{q^\alpha}{|\alpha|!}$, *satisfy*

$$\sum_k q_k \lambda_k^2 \left(C_A^{ellip}\right)^2 < \frac{1}{2}, \quad \text{and} \quad \sum_k \frac{q_k}{\rho'_k} < \frac{1}{2}. \tag{4.6}$$

Here, the constant C *is independent of* h, M, n, *and the constant* $C_{M,n}$ *can be taken as*

$$C_{M,n} = C' \binom{M+n}{M}$$

where C' *is independent of* h, M, n. *The term*

$$Q_{M,n}(\mathcal{R}, \mathcal{R}') = \sqrt{\frac{\hat{Q}_W}{(1-\hat{Q})^2} + \frac{\hat{Q}^{n+1}}{1-\hat{Q}}}$$

where

$$\hat{Q} = \sum_{k \geq 1} q_k \lambda_k^2 \left(C_A^{ellip}\right)^2 + \frac{q_k}{\rho'_k} < 1 \quad \text{and} \quad \hat{Q}_W = \sum_{k > M} q_k \lambda_k^2 \left(C_A^{ellip}\right)^2 + \frac{q_k}{\rho'_k}.$$

The proof of Theorem 4.1 is deferred to Section 6. We proceed to make some remarks.

In analogy to the deterministic equation case, the finite element convergence rate of h^{m+1} for the solution $u \in \mathcal{R}_\Omega H_T^1 H_X^m$ is optimal. Without invoking the stochastic adjoint problem, it is easy to obtain a finite element convergence rate of h^{m-1} for the solution $u \in \mathcal{R}_\Omega H_T^1 H_X^m$, which is two orders worse than optimal. The gain of two orders is achieved by extracting some crucial information from the estimates of lower norms, through the application of the stochastic adjoint problem in the duality technique.

The term $Q_{M,n}(\mathcal{R}, \mathcal{R}')$ in the estimate (4.5) is, as usual, the error from truncating the Wiener chaos expansion up to $\mathcal{J}_{M,n}$. It arises from invoking the error estimates for the corresponding elliptic problem, and depends on the *choice* of the weighted space $\mathcal{R}$ in which to bound the error, as well as on the weights $\mathcal{R}'$ of the forcing term in the sense of the *elliptic problem*. It also implicitly assumes that $\mathcal{R}, \mathcal{R}'$ are related by the condition (4.6). However, the second inequality in (4.6) is a somewhat strict condition. If we consider the optimal weights $\mathcal{R}'$ to behave like $\rho'_k \sim k^{-(1+\epsilon)} \lambda_k^{-2}$ for any $\epsilon > 0$, then the optimal weights $\mathcal{R}$ can behave like $q_k \sim k^{-(2+\epsilon)} \lambda_k^{-2}$ for any $\epsilon > 0$. Thus, the error estimate holds in a weighted space that is generally worse than the optimal space that the solution u belongs to. Additionally, the validity of the first and third term in the RHS of (4.5) requires the boundedness of u_{tt} in the H_X^{-1} norm. This marks the departure of the SFEM from the deterministic FEM.

Since the proof of Theorem 4.1 makes heavy use of the SFEM error estimates for the corresponding stochastic elliptic problem as well as the stochastic adjoint problem, we will devote the next section to addressing these two issues.

5. The Corresponding Stochastic Elliptic Problem

The corresponding stochastic elliptic problem is

$$\begin{aligned} \mathcal{A}U + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}U) &= F \quad \text{in } D \\ U|_{\partial D} &= 0 \end{aligned} \tag{5.1}$$

where $F \in \bar{\mathcal{R}}_\Omega H_X^{-1}$. For non-random F, [10] has shown the unique existence of the weak solution U in some $\mathcal{R}_\Omega H_{0X}^1$. For arbitrary random F, an argument identical to Theorem 3.3 implies that U belongs to $\mathcal{R}_\Omega H_{0X}^1$, provided the weights $r_\alpha^2 = \frac{q^\alpha}{|\alpha|!}$ satisfies

$$\sum_k q_k C_k^2 < 1, \quad \text{and} \quad \sum_k \frac{q_k}{\bar{\rho}_k} < 1, \tag{5.2}$$

where $C_k = C_A^{ellip}\lambda_k$ are the constants defined by $\|\boldsymbol{A}^{-1}\boldsymbol{M}_k v\|_{H_{0X}^1} \le C_k \|v\|_{H_{0X}^1}$ for all $v \in H_{0X}^1$.

We first state a result on the boundedness of the stochastic operator in the LHS of equation (5.1) that will come in handy subsequently.

Lemma 5.1. *Let $\chi \in \mathcal{R}_\Omega H_{0X}^1$, where the weights $\mathcal{R}$ satisfy $\sum_k q_k \lambda_k^2 < \infty$. Then there exists C depending only on $\mathcal{R}, \mathcal{A}, \mathcal{M}$ such that*

$$\|\mathcal{A}\chi + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}\chi)\|_{\mathcal{R}_\Omega H_X^{-1}} \le C\|\chi\|_{\mathcal{R}_\Omega H_{0X}^1}.$$

Proof. By direct computation,

$$\begin{aligned} &\|\mathcal{A}\chi + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}\chi)\|^2_{\mathcal{R}_\Omega H_X^{-1}} = \sum_\alpha r_\alpha^2 \|\mathcal{A}\chi_\alpha + \sum_{k=1}^\infty \sqrt{\alpha_k}\mathcal{M}_k \chi_{\alpha-\varepsilon_k}\|^2_{H_X^{-1}} \\ &\le \sum_\alpha r_\alpha^2 \left(C_A^b \|\chi\|_{H_0^1} + \sum_{k=1}^\infty \sqrt{\alpha_k}\lambda_k \|\chi_{\alpha-\varepsilon_k}\|_{H_0^1} \right)^2 \\ &\le 2(C_A^b)^2 \|\chi\|^2_{\mathcal{R}_\Omega H_{0X}^1} + 2 \underbrace{\sum_\alpha r_\alpha^2 \left(\sum_{k=1}^\infty \sqrt{\alpha_k}\lambda_k \|\chi_{\alpha-\varepsilon_k}\|_{H_0^1} \right)^2}_{(*)} \end{aligned}$$

where C_A^b is the constant in $\|\mathcal{A}\phi\|_{H_X^{-1}} \le C_A^b \|\phi\|_{H^1}$, for all $\phi \in H_{0X}^1$. To estimate $(*)$, we apply Jensen's inequality to obtain

$$\begin{aligned} (*) &= \sum_\alpha r_\alpha^2 \left(\sum_{\substack{k=1 \\ \alpha_k \neq 0}}^\infty \frac{\alpha_k}{|\alpha|} \frac{|\alpha|}{\sqrt{\alpha_k}} \lambda_k \|\chi_{\alpha-\varepsilon_k}\|_{H_0^1} \right)^2 \\ &\le \sum_\alpha \frac{q^\alpha}{|\alpha|!} \sum_{\substack{k=1 \\ \alpha_k \neq 0}}^\infty \frac{\alpha_k}{|\alpha|} \frac{|\alpha|^2}{\alpha_k} \lambda_k^2 \|\chi_{\alpha-\varepsilon_k}\|^2_{H_0^1} \\ &= \sum_\alpha \sum_k \mathbf{1}_{\{\alpha_k \neq 0\}} q_k \lambda_k^2 \frac{q^{\alpha-\varepsilon_k}}{(|\alpha|-1)!} \|\chi_{\alpha-\varepsilon_k}\|^2_{H_0^1} \end{aligned}$$

$$= \sum_k q_k \lambda_k^2 \sum_{\substack{\alpha \\ \alpha_k \neq 0}} r_{\alpha - \varepsilon_k} \|\chi_{\alpha-\varepsilon_k}\|^2_{H_0^1} = \left(\sum_k q_k \lambda_k^2 \right) \|\chi\|^2_{\mathcal{R}_\Omega H^1_{0X}}$$

Hence,

$$\|\mathcal{A}\chi + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}\chi)\|^2_{\mathcal{R}_\Omega H_X^{-1}} \leq 2 \left((C_A^b)^2 + \sum_k q_k \lambda_k^2 \right) \|\chi\|^2_{\mathcal{R}_\Omega H^1_{0X}}.$$

□

5.1. The formal stochastic adjoint problem. In this section, we study the formal stochastic adjoint problem,

$$\mathcal{A}^*\psi + \mathcal{M}^* \cdot \boldsymbol{D}_{\dot{W}}\psi = \phi \quad \text{on } D \qquad (5.3)$$
$$\psi|_{\partial D} = 0$$

where $\phi \in \mathcal{R}_\Omega^{-1} H_X^{-1}$. The operators $\mathcal{A}^*, \mathcal{M}_k^*$ are the formal adjoints of $\mathcal{A}, \mathcal{M}_k$, respectively. By definition, the term $\mathcal{M}^* \cdot \boldsymbol{D}_{\dot{W}}\psi$ can be formally written as

$$(\mathcal{M}^* \cdot \boldsymbol{D}_{\dot{W}}\psi)_\alpha = \sum_{k=1}^{\infty} \sqrt{\alpha_k + 1} \mathcal{M}_k^* \psi_{\alpha+\varepsilon_k}, \quad \text{for } \alpha \in \mathcal{J}$$

where the infinite sum is interpreted as convergent in an appropriate space.

Definition 5.2. A *weak solution* of (5.3), with $\phi \in \mathcal{R}_\Omega^{-1} H_X^{-1}$, is a process $\psi \in \mathcal{R}_\Omega^{-1} H^1_{0X}$ such that

$$\langle\langle \chi, \mathcal{A}^*\psi + \mathcal{M}^* \cdot \boldsymbol{D}_{\dot{W}}\psi \rangle\rangle_{\mathcal{R}_\Omega H^1_{0X}, \mathcal{R}_\Omega^{-1} H_X^{-1}} = \langle\langle \chi, \phi \rangle\rangle_{\mathcal{R}_\Omega H^1_{0X}, \mathcal{R}_\Omega^{-1} H_X^{-1}}$$

for all $\chi \in \mathcal{R}_\Omega H^1_{0X}$.

Since $\boldsymbol{D}_{\dot{W}}$ and $\boldsymbol{\delta}_{\dot{W}}$ are adjoint to each other,

$$\langle\langle \chi, \mathcal{A}^*\psi + \mathcal{M}^* \cdot \boldsymbol{D}_{\dot{W}}\psi \rangle\rangle_{\mathcal{R}_\Omega H^1_{0X}, \mathcal{R}_\Omega^{-1} H_X^{-1}} = \langle\langle \mathcal{A}\chi + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}\chi), \psi \rangle\rangle_{\mathcal{R}_\Omega H_X^{-1}, \mathcal{R}_\Omega^{-1} H^1_{0X}}.$$

Denote by C_A^* the constant in $\|U\|_{H_0^1} \leq C_A^* \|F\|_{H^{-1}}$ for the solution of $\mathcal{A}^*U = F$. Denote by λ_k^* the constant in $\|\mathcal{M}_k^*\phi\|_{H^{-1}} \leq \lambda_k^* \|\phi\|_{H_0^1}$. For brevity, in this section only, we may drop the superscripts $*$ and write C_A, λ_k without ambiguity.

Proposition 5.3. *Suppose there exists $\{\psi_\alpha, \alpha \in \mathcal{J}\}$ belonging to H_0^1 such that for all α,*

(i) $\sum_{k=1}^\infty \sqrt{\alpha_k + 1} \mathcal{M}_k^* \psi_{\alpha+\varepsilon_k} \in H_X^{-1}$;

(ii) $\mathcal{A}^*\psi_\alpha + \sum_{k=1}^\infty \sqrt{\alpha_k + 1} \mathcal{M}_k^* \psi_{\alpha+\varepsilon_k} = \phi_\alpha$ *in the weak sense.*

Let the weights $\mathcal{R}$ satisfy $\sum_k q_k (\lambda_k^* C_A^*)^2 < \frac{1}{2}$. Then there exists C depending on $\mathcal{R}, \mathcal{A}^*, \mathcal{M}^*$, such that

$$\|\psi\|_{\mathcal{R}_\Omega^{-1} H^1_{0X}} \leq C \|\phi\|_{\mathcal{R}_\Omega^{-1} H_X^{-1}}.$$

Proof. From the deterministic elliptic estimates,

$$\|\psi_\alpha\|_{H^1_{0X}} \leq C_A \left(\|\phi_\alpha\|_{H^{-1}} + \sum_k \sqrt{\alpha_k + 1} \|\mathcal{M}_k^* \psi_{\alpha+\varepsilon_k}\|_{H^{-1}} \right)$$

So

$$\sum_\alpha r_\alpha^{-2}\|\psi_\alpha\|^2_{H^1_{0X}} \le 2C_A^2 \sum_\alpha r_\alpha^{-2}\|\phi_\alpha\|^2_{H^{-1}} + 2\sum_\alpha C_A^2\left(\sum_k r_\alpha^{-1}\sqrt{\alpha_k+1}\lambda_k\|\psi_{\alpha+\varepsilon_k}\|_{H^1_{0X}}\right)^2$$

The second term can be estimated by

$$\begin{aligned}
&C_A^2\left(\sum_k r_\alpha^{-1}\sqrt{\alpha_k+1}\lambda_k\|\psi_{\alpha+\varepsilon_k}\|_{H^1_{0X}}\right)^2\\
&=\left(\sum_k \frac{\sqrt{|\alpha|!}}{q^{\alpha/2}}\frac{\sqrt{|\alpha|+1}}{q_k^{1/2}}\sqrt{\frac{\alpha_k+1}{|\alpha|+1}}q_k^{1/2}\lambda_k C_A\|\psi_{\alpha+\varepsilon_k}\|_{H^1_0}\right)^2\\
&\le\left(\sum_k r_{\alpha+\varepsilon_k}^{-2}\|\psi_{\alpha+\varepsilon_k}\|^2_{H^1_0}\frac{\alpha_k+1}{|\alpha|+1}\right)\left(\sum_k q_k\lambda_k^2C_A^2\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_\alpha\sum_k r_{\alpha+\varepsilon_k}^{-2}\|\psi_{\alpha+\varepsilon_k}\|^2_{H^1_0}\frac{\alpha_k+1}{|\alpha|+1} = \sum_k\sum_{\beta:\beta_k\neq 0} r_\beta^{-2}\|\psi_\beta\|^2_{H^1_0}\frac{\beta_k}{|\beta|}\\
&=\sum_\beta\sum_k\frac{\beta_k}{|\beta|}r_\beta^{-2}\|\psi_\beta\|^2_{H^1_0} = \|\psi\|^2_{\mathcal{R}_\Omega^{-1}H^1_{0X}}
\end{aligned}$$

Hence,

$$\left(1-2\big(\sum_k q_k\lambda_k^2C_A^2\big)\right)\|\psi\|^2_{\mathcal{R}_\Omega^{-1}H^1_{0X}} \le 2C_A^2\|\phi\|^2_{\mathcal{R}_\Omega^{-1}H^{-1}_X}.$$

The estimate follows from the condition (5.4). □

Theorem 5.4. *There exists a weak solution $\psi \in \mathcal{R}_\Omega^{-1}H^1_{0X}$ to the adjoint problem* (5.3)*, provided*

$$\sum_k q_k(\lambda_k^* C_A^*)^2 < \frac{1}{2}. \tag{5.4}$$

Proof. The weak solution is constructed via the usual Galerkin approach. Let $\phi^p := \sum_{|\alpha|\le p}\phi_\alpha\xi_\alpha$. We will first construct the weak solution ψ^p of

$$\mathcal{A}^*\psi^p + \mathcal{M}^*\cdot \boldsymbol{D}_{\dot W}\psi^p = \phi^p. \tag{5.5}$$

Let $\psi^p_\alpha = 0$ if $|\alpha| > p$. For $|\alpha| = p$, define ψ^p_α by the solution of

$$\mathcal{A}^*\psi^p_\alpha = \phi_\alpha,$$

and for $|\alpha| < p$,

$$\mathcal{A}^*\psi^p_\alpha = \phi_\alpha - \sum_k \sqrt{\alpha_k+1}\mathcal{M}_k^*\psi^p_{\alpha+\varepsilon_k}.$$

The solvability of the equation for $|\alpha| = p$ follows from the usual deterministic theory, and

$$\|\psi^p_\alpha\|_{H^1_0} \le C_A\|\phi_\alpha\|_{H^{-1}}.$$

The solvability of the equation for $|\alpha| < p$ requires that $\sum_k \sqrt{\alpha_k + 1}\mathcal{M}_k^* \psi^p_{\alpha+\varepsilon_k}$ belongs to H_X^{-1}, which we now verify.

Denote by $\Phi_\alpha^{(i)}$ the quantity

$$(\Phi_\alpha^{(i)})^2 = \sum_{k_1,\ldots,k_i=1}^{\infty} r^{-2}_{\alpha+\varepsilon_{k_1}+\cdots+\varepsilon_{k_i}} \|\phi_{\alpha+\varepsilon_{k_1}+\cdots+\varepsilon_{k_i}}\|^2_{H^{-1}} \times \prod_{j=1}^{i} \frac{(\alpha+\varepsilon_{k_1}+\cdots+\varepsilon_{k_{j-1}})_{k_j}+1}{|\alpha|+j}$$

Clearly, $\Phi_\alpha^{(i)} < \infty$. If $|\alpha| = p - l$, for $l = 1, \ldots, p$, it is easy to show by induction on l that

$$\|\psi_\alpha^p\|_{H_0^1} \le C_A \left(\|\phi_\alpha\|_{H^{-1}} + r_\alpha^{-1}\sqrt{(l-1)!} \sum_{i=1}^{l} 2^{i/2}\hat{q}^{i/2}\Phi_\alpha^{(i)} \right)$$

where $\hat{q} = \sum_k q_k \lambda_k^2 C_A^2$, and hence

$$r_\alpha^{-2} \| \sum_k \sqrt{\alpha_k+1}\mathcal{M}_k^* \psi^p_{\alpha+\varepsilon_k}\|^2_{H^{-1}} \le (l-1)! \sum_{i=1}^{l} 2^i \hat{q}^i \Phi_\alpha^{(i)} < \infty.$$

This verifies that $\sum_k \sqrt{\alpha_k+1}\mathcal{M}_k^*\psi^p_{\alpha+\varepsilon_k} \in H^{-1}$, and hence $\psi^p := \sum_\alpha \psi_\alpha^p \xi_\alpha$ is well-defined.

By construction, ψ^p solves equation (5.5). Moreover, by similar calculations as Proposition 5.3,

$$\left(1 - 2\big(\sum_k q_k \lambda_k^2 C_A^2\big)\right) \|\psi^p\|_{\mathcal{R}_\Omega^{-1} H^1_{0X}} \le 2C_A \|\phi^p\|_{\mathcal{R}_\Omega^{-1} H_X^{-1}} \le 2C_A\|\phi\|_{\mathcal{R}_\Omega^{-1}H_X^{-1}}$$

and by (5.4), the sequence ψ^p is uniformly bounded in $\mathcal{R}_\Omega^{-1}H^1_{0X}$. Thus, there exists a weakly converging subsequence, say, with abuse of notation, $\psi^p \rightharpoonup \psi$ weakly in $\mathcal{R}_\Omega^{-1}H^1_{0X}$.

Fix an arbitrary $\chi \in \mathcal{R}_\Omega H^1_{0X}$, and from Lemma 5.1, $\mathcal{A}\chi + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}\chi) =: F$ belongs to $\mathcal{R}_\Omega H_X^{-1}$. Then

$$\begin{aligned} \langle\langle \mathcal{A}^*\psi + \mathcal{M}^* \cdot \boldsymbol{D}_{\dot{W}}\psi,\, \chi\rangle\rangle &= \langle\langle \psi,\, \mathcal{A}\chi + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}\chi)\rangle\rangle = \lim_{p\to\infty} \langle\langle \psi^p, F\rangle\rangle \\ &= \lim_{p\to\infty} \langle\langle \mathcal{A}^*\psi^p + \mathcal{M}^*\cdot \boldsymbol{D}_{\dot{W}}\psi^p,\, \chi\rangle\rangle = \lim_{p\to\infty}\langle\langle \phi^p, \chi\rangle\rangle = \langle\langle\phi,\chi\rangle\rangle. \end{aligned}$$

□

By definition, the solution in Theorem 5.4 satisfies the hypothesis of Proposition 5.3.

Remark 5.5. Higher spatial regularity results follow as usual from the corresponding deterministic results for each equation in the propagator. In a similar fashion to the proof of Theorem 5.4, one can obtain higher regularity estimates such as

$$\|\psi\|_{\mathcal{R}_\Omega^{-1}H_X^r} \le C\|\phi\|_{\mathcal{R}_\Omega^{-1}H_X^{r-2}}$$

for $r \ge 1$, if $\phi \in \mathcal{R}_\Omega^{-1}H_X^{r-2}$, and if the boundary ∂D and the coefficients of $\mathcal{A}, \mathcal{M}_k$ are sufficiently smooth.

5.2. SFEM for the stochastic elliptic problem. An extension of [14] to random forcing terms yields the following result for the approximation error of the SFEM approximation $U_h^{M,n}$ of equation (5.1).

Theorem 5.6. *Suppose $U \in \mathcal{R}_\Omega H^1_{0X} \cap \mathcal{R}_\Omega H^{m+1}_X$, where there weights satisfy*

$$\sum_k q_k C_k^2 < \frac{1}{2}, \quad \text{and} \quad \sum_k \frac{q_k}{\bar\rho_k} < \frac{1}{2}, \tag{5.6}$$

Then the error of approximation of the stochastic finite element method is given by

$$\begin{aligned} &\|U - U_h^{M,n}\|_{\mathcal{R}_\Omega H^1_{0X}} \\ &\le C_{M,n} h^m \|U\|_{\mathcal{R}_\Omega H^{m+1}_X} + C\|F\|_{\bar{\mathcal{R}}_\Omega H^{-1}_X} Q_{M,n}(\mathcal{R}, \bar{\mathcal{R}}) \end{aligned} \tag{5.7}$$

where $C_{M,n} = C'\binom{M+n}{M}$, and the constants C, C' are independent of h, M, n.

The proof of Theorem 5.6 will be given in Appendix A. We will also need error estimates in lower norms.

Proposition 5.7. *Under the same assumptions as Theorem 5.6, the error of approximation of the SFEM has the bounds*

$$\begin{aligned} &\|U - U_h^{M,n}\|_{\mathcal{R}_\Omega H^{1-k}_X} \\ &\le C_{M,n} h^{m+k} \|U\|_{\mathcal{R}_\Omega H^{m+1}_X} + C\|F\|_{\bar{\mathcal{R}}_\Omega H^{-1}_X} Q_{M,n}(\mathcal{R}, \bar{\mathcal{R}}) \end{aligned} \tag{5.8}$$

for $k = 1, 2$.

Proof. As in the proof of Theorem 5.6,

$$U - U_h^{M,p} = \sum_{\alpha \in \mathcal{J}_{M,p}} (U_\alpha - \hat U_\alpha)\xi_\alpha + \sum_{\alpha \in \mathcal{J} \setminus \mathcal{J}_{M,p}} U_\alpha \xi_\alpha =: e_1 + e_2,$$

with

$$\begin{aligned} \|e_1\|_{\mathcal{R}_\Omega H^1_{0X}} &\le C_{M,n} h^m \|U\|_{\mathcal{R}_\Omega H^{m+1}_X}, \quad \text{and} \\ \|e_2\|_{\mathcal{R}_\Omega H^1_{0X}} &\le C\|F\|_{\bar{\mathcal{R}}_\Omega H^{-1}_X} Q_{M,n}(\mathcal{R}, \bar{\mathcal{R}}) \end{aligned}$$

We leave the estimate for e_2 untouched. For e_1, we consider the two cases.

Case: $k = 1$. Let $\psi \in \mathcal{R}_\Omega^{-1} H^2_X$ be the solution of $\mathcal{A}\psi + \mathcal{M} \cdot \boldsymbol{D}_{\dot W}\psi = \mathcal{R}^2 e_1$, with $\|\psi\|_{\mathcal{R}_\Omega^{-1} H^2_X} \le C\|\mathcal{R}^2 e_1\|_{\mathcal{R}_\Omega^{-1} L^2_X} = \|e_1\|_{\mathcal{R}_\Omega L^2_X}$. Note that, in fact, $\psi \in (S^{M,n})^* \otimes H^3_X$ also. Then,

$$\begin{aligned} \|e_1\|^2_{\mathcal{R}_\Omega L^2_X} &= \langle\langle e_1, \mathcal{R}^2 e_1\rangle\rangle_{\mathcal{R}_\Omega L^2_X, \mathcal{R}_\Omega^{-1} L^2_X} = \langle\langle e_1, \mathcal{R}^2 e_1\rangle\rangle_{\mathcal{R}_\Omega H^{-1}_X, \mathcal{R}_\Omega^{-1} H^1_X} \\ &= \langle\langle e_1, \mathcal{A}\psi + \mathcal{M} \cdot \boldsymbol{D}_{\dot W}\psi\rangle\rangle_{\mathcal{R}_\Omega H^{-1}_X, \mathcal{R}_\Omega^{-1} H^1_X} \\ &= \langle\langle \mathcal{A}e_1 + \boldsymbol{\delta}_{\dot W}(\mathcal{M}e_1),\, \psi - \chi\rangle\rangle_{\mathcal{R}_\Omega H^{-1}_X, \mathcal{R}_\Omega^{-1} H^1_X} \end{aligned}$$

for all $\chi \in S^{M,n} \otimes S_h$. So

$$\|e_1\|^2_{\mathcal{R}_\Omega L^2_X} \le \|\mathcal{A}e_1 + \boldsymbol{\delta}_{\dot W}(\mathcal{M}e_1)\|_{\mathcal{R}_\Omega H^{-1}_X} \inf_{\chi \in S^{M,n} \otimes S_h} \|\psi - \chi\|_{\mathcal{R}_\Omega^{-1} H^1_X}$$

To estimate the first term, Lemma 5.1 implies that

$$\|\mathcal{A}e_1 + \boldsymbol{\delta}_{\dot W}(\mathcal{M}e_1)\|_{\mathcal{R}_\Omega H^{-1}_X} \le C\|e_1\|_{\mathcal{R}_\Omega H^1_{0X}}$$

To estimate the second term, we make use of the FE estimate (4.1), in particular

$$\inf_{\chi_h \in S_h} \|\Phi - \chi_h\|_{H^1_{0X}} \le Ch\|\Phi\|_{H^2_X}, \quad \forall \Phi \in H^2_X \cap H^1_{0X}.$$

This FE estimate is usually obtained by finding a projection operator I_h for which $\|\Phi - I_h\Phi\|_{H^1_{0X}} \le Ch^2\|\Phi\|_{H^3_X}$, from which the desired estimate follows immediately. But here, we will show the estimate by constructing a near-infimizing χ. Fix $\epsilon > 0$. For each $\alpha \in \mathcal{J}_{M,n}$, there exists $\chi_\alpha \in S_h$ such that

$$\|\psi_\alpha - \chi_\alpha\|_{H^1_{0X}} \le \inf_{\chi_h \in S_h} \|\psi_\alpha - \chi_h\|_{H^1_{0X}} + \kappa_\alpha(\epsilon) \le Ch\|\psi_\alpha\|_{H^2_X} + \kappa_\alpha(\epsilon)$$

where we choose $\kappa_\alpha(\epsilon) = \epsilon^{1/2} r_\alpha \bar{\kappa}_\alpha$, with $\sum_\alpha \bar{\kappa}_\alpha^2 = \frac{1}{2}$. Set $\chi = \sum_{\alpha \in \mathcal{J}_{M,n}} \chi_\alpha \xi_\alpha \in S^{M,n} \otimes S_h$. Then

$$\|\psi - \chi\|^2_{\mathcal{R}_\Omega^{-1} H^1_{0X}} \le \sum_{\alpha \in \mathcal{J}_{M,n}} r_\alpha^{-2} \Big(Ch\|\psi_\alpha\|_{H^2_X} + \kappa_\alpha(\epsilon) \Big)^2 \le Ch^2\|\psi\|^2_{\mathcal{R}_\Omega^{-1} H^2_X} + \epsilon$$

and

$$\inf_{\chi \in S^{M,n} \otimes S_h} \|\psi - \chi\|_{\mathcal{R}_\Omega^{-1} H^1_{0X}} \le Ch\|\psi\|_{\mathcal{R}_\Omega^{-1} H^2_X} \le Ch\|e_1\|_{\mathcal{R}_\Omega H^1_{0X}}$$

Hence,

$$\begin{aligned} \|e_1\|^2_{\mathcal{R}_\Omega L^2_X} &\le \|e_1\|_{\mathcal{R}_\Omega H^1_{0X}} Ch\|\psi\|_{\mathcal{R}_\Omega^{-1} H^2_X} \\ &\le C_{M,n} h^{m+1} \|U\|_{\mathcal{R}_\Omega H^{m+1}_X} \|e_1\|_{\mathcal{R}_\Omega L^2_X}. \end{aligned}$$

Case: $k = 2$. Since $e_1 \in S^{M,n} \otimes H^1_{0X}$, we compute the norm

$$\|e_1\|_{\mathcal{R}_\Omega H_X^{-1}} = \sup_{\phi \in \mathcal{R}_\Omega^{-1} H^1_{0X}} \frac{|\langle\langle e_1, \phi \rangle\rangle|}{\|\phi\|_{\mathcal{R}_\Omega^{-1} H^1_{0X}}} = \sup_{\phi \in (S^{M,n})^* \otimes H^1_{0X}} \frac{|\langle\langle e_1, \phi \rangle\rangle|}{\|\phi\|_{\mathcal{R}_\Omega^{-1} H^1_{0X}}}$$

For any $\phi \in (S^{M,n})^* \otimes H^1_{0X}$, let $\psi \in \mathcal{R}_\Omega^{-1} H^3_X$ be the solution of $\mathcal{A}\psi + \mathcal{M} \cdot \boldsymbol{D}_{\dot{W}} \psi = \phi$, with $\|\psi\|_{\mathcal{R}_\Omega^{-1} H^3_X} \le C\|\phi\|_{\mathcal{R}_\Omega^{-1} H^1_{0X}}$. Note that, in fact, $\psi \in (S^{M,n})^* \otimes H^3_X$ also. Then,

$$\langle\langle e_1, \phi \rangle\rangle = \langle\langle e_1, \mathcal{A}\psi + \mathcal{M} \cdot \boldsymbol{D}_{\dot{W}} \psi \rangle\rangle = \langle\langle \mathcal{A}e_1 + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}e_1),\, \psi - \chi \rangle\rangle$$

for all $\chi \in S^{M,n} \otimes S_h$, and by a similar argument in the previous case, we have that

$$\begin{aligned} |\langle\langle e_1, \phi \rangle\rangle| &\le \|\mathcal{A}e_1 + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}e_1)\|_{\mathcal{R}_\Omega H_X^{-1}} \inf_{\chi \in S^{M,n} \otimes S_h} \|\psi - \chi\|_{\mathcal{R}_\Omega^{-1} H^1_{0X}} \\ &\le Ch^2 \|e_1\|_{\mathcal{R}_\Omega H^1_{0X}} \|\phi\|_{\mathcal{R}_\Omega^{-1} H^1_{0X}} \\ &\le C_{M,n} h^{m+2} \|U\|_{\mathcal{R}_\Omega H^{m+1}} \|\phi\|_{\mathcal{R}_\Omega^{-1} H^1_{0X}}. \end{aligned}$$

The result follows. □

6. Proof of Theorem 4.1

Let $\Pi_h^{M,n}$ denote the SFEM approximation operator for the stochastic *elliptic* problem (5.1). In particular,

$$\Big\langle\Big\langle \mathcal{A}U + \sum_{k=1}^{M} \boldsymbol{\delta}_{\xi_k}(\mathcal{M}_k U), z \Big\rangle\Big\rangle = \Big\langle\Big\langle \mathcal{A}(\Pi_h^{M,n} U) + \sum_{k=1}^{M} \boldsymbol{\delta}_{\xi_k}(\mathcal{M}_k(\Pi_h^{M,n} U)), z \Big\rangle\Big\rangle$$

for all $z \in S^{M,n} \otimes S_h$. The error estimates (5.7) also imply that $\Pi_h^{M,n}$ is a continuous linear map from $\mathcal{R}_\Omega H^1_{0X}$ into itself.

Decompose the error into

$$\begin{aligned} e_h(t) := u_h^{M,n}(t) - u(t) &= \left(u_h^{M,n}(t) - \Pi_h^{M,n} u(t)\right) + \left(\Pi_h^{M,n} u(t) - u(t)\right) \\ &= \theta(t) + \pi(t). \end{aligned}$$

Analysis for π. For every $t \in (0,T]$, we have that $\mathcal{A}u(t) + \boldsymbol{\delta}_{\dot{W}}(\mathcal{M}u(t)) = f(t) - u_t(t) \in \mathcal{R}'_\Omega H_X^{m-1}$. Hence the elliptic estimates (5.7) and lower norm estimates (5.8) imply

$$\begin{aligned} &\|\pi(t)\|_{\mathcal{R}_\Omega L^2_X} = \|\Pi_h^{M,n} u(t) - u(t)\|_{\mathcal{R}_\Omega L^2_X} \\ &\leq C_{M,n} h^{m+1} \|u(t)\|_{\mathcal{R}_\Omega H_X^{m+1}} + C\|f(t) - u_t(t)\|_{\mathcal{R}'_\Omega H_X^{-1}} Q_{M,n}(\mathcal{R},\mathcal{R}') \end{aligned}$$

provided (4.6) holds.

Analysis for θ. From the definitions of the numerical and weak solutions,

$$\begin{aligned} &\langle\langle \theta_t, z\rangle\rangle + \Big\langle\Big\langle \mathcal{A}\theta + \sum_{k=1}^M \boldsymbol{\delta}_{\xi_k}(\mathcal{M}_k\theta), z\Big\rangle\Big\rangle \\ &= \langle\langle f, z\rangle\rangle - \Big\langle\Big\langle (\Pi_h^{M,n} u)_t, z\Big\rangle\Big\rangle - \Big\langle\Big\langle \mathcal{A}\Pi_h^{M,n} u + \sum_{k=1}^M \boldsymbol{\delta}_{\xi_k}(\mathcal{M}_k(\Pi_h^{M,n} u)), z\Big\rangle\Big\rangle \\ &= \langle\langle f, z\rangle\rangle - \Big\langle\Big\langle (\Pi_h^{M,n} u)_t, z\Big\rangle\Big\rangle - \Big\langle\Big\langle \mathcal{A}u + \sum_{k=1}^M \boldsymbol{\delta}_{\xi_k}(\mathcal{M}_k u), z\Big\rangle\Big\rangle \pm \langle\langle u_t, z\rangle\rangle \\ &= -\Big\langle\Big\langle (\Pi_h^{M,n} u - u)_t, z\Big\rangle\Big\rangle \end{aligned}$$

for all $z \in S^{M,n} \otimes S_h$. Choosing $z = \mathcal{R}^2\theta$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{\mathcal{R}_\Omega L^2_X} + \sum_{\alpha\in\mathcal{J}_{M,n}} r_\alpha^2 \boldsymbol{A}[\theta_\alpha,\theta_\alpha] \\ &\leq \|(\Pi_h^{M,n} u - u)_t\|_{\mathcal{R}_\Omega H_X^{-1}} \|\theta\|_{\mathcal{R}_\Omega H^1_{0X}} \\ &\qquad + \sum_{\alpha\in\mathcal{J}_{M,n}} \sum_{k=1}^M \sqrt{\alpha_k}\lambda_k r_\alpha^2 \|\theta_{\alpha-\varepsilon_k}\|_{H^1_{0X}} \|\theta_\alpha\|_{H^1_{0X}} = (I) + (II) \end{aligned}$$

where we recall that λ_k are the constants in $\boldsymbol{M}_k[u,v] \leq \lambda_k \|u\|_{H^1_{0X}} \|v\|_{H^1_{0X}}$.

For (II),

$$\begin{aligned} (II) &= \sum_{\alpha\in\mathcal{J}_{M,n}} \sum_{k=1}^M \sqrt{\alpha_k}\lambda_k r_\alpha \|\theta_{\alpha-\varepsilon_k}\|_{H^1_X} r_\alpha \|\theta_\alpha\|_{H^1_X} \\ &\leq \left(\sum_{\alpha\in\mathcal{J}_{M,n}} \left(\sum_{k=1}^M \sqrt{\alpha_k}\lambda_k r_\alpha \|\theta_{\alpha-\varepsilon_k}\|_{H^1_X}\right)^2\right)^{1/2} \left(\sum_{\alpha\in\mathcal{J}_{M,n}} r_\alpha^2 \|\theta_\alpha\|^2_{H^1_X}\right)^{1/2} \end{aligned}$$

$$\leq \left(\sum_{\alpha \in \mathcal{J}_{M,n}} \left(\sum_{\substack{k=1 \\ \alpha_k \neq 0}}^{M} \frac{\alpha_k}{|\alpha|} \sqrt{\frac{|\alpha|}{\alpha_k}} \lambda_k q_k^{1/2} r_{\alpha-\varepsilon_k} \|\theta_{\alpha-\varepsilon_k}\|_{H^1_X} \right)^2 \right)^{1/2} \|\theta\|_{\mathcal{R}_\Omega H^1_X}$$

$$\leq \left(\sum_{\alpha \in \mathcal{J}_{M,n}} \sum_{\substack{k=1 \\ \alpha_k \neq 0}}^{M} \frac{\alpha_k}{|\alpha|} \left(\sqrt{\frac{|\alpha|}{\alpha_k}} \lambda_k q_k^{1/2} r_{\alpha-\varepsilon_k} \|\theta_{\alpha-\varepsilon_k}\|_{H^1_X} \right)^2 \right)^{1/2} \|\theta\|_{\mathcal{R}_\Omega H^1_X}$$

where we applied Jensen's inequality in the last inequality. Continuing,

$$(II) \leq \left(\sum_{k=1}^{M} \sum_{\substack{\alpha \in \mathcal{J}_{M,n} \\ \alpha_k \neq 0}} \lambda_k^2 q_k r^2_{\alpha-\varepsilon_k} \|\theta_{\alpha-\varepsilon_k}\|^2_{H^1} \right)^{1/2} \|\theta\|_{\mathcal{R}_\Omega H^1_X}$$

$$\leq \left(\sum_{k=1}^{M} \lambda_k^2 q_k \right)^{1/2} \|\theta\|^2_{\mathcal{R}_\Omega H^1_X} := \overline{[q\vec{\lambda}^2]}^{1/2}_{\leq M} \|\theta\|^2_{\mathcal{R}_\Omega H^1_X}$$

Then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{\mathcal{R}_\Omega L^2_X} + C_A^{coerc}\|\theta\|^2_{\mathcal{R}_\Omega H^1_{0X}} \\ &\leq \epsilon_0 \|(\Pi_h^{M,n}u - u)_t\|^2_{\mathcal{R}_\Omega H^{-1}_X} + \left(\frac{1}{4\epsilon_0} + \overline{[q\vec{\lambda}^2]}^{1/2}_{\leq M} \right) \|\theta\|^2_{\mathcal{R}_\Omega H^1_{0X}} \end{aligned}$$

where C_A^{coerc} is the coercivity constant in $\boldsymbol{A}[u,u] \geq C_A^{coerc}\|u\|^2_{H^1_0}$ for all $u \in H^1_0$, and we have that $C_A^{coerc} = (C_A^{ellip})^{-1}$. By the first condition in (4.6), we can find ϵ_0 such that $\frac{1}{4\epsilon_0} + \overline{[q\vec{\lambda}^2]}^{1/2}_{\leq M} = C_A^{coerc}$. So

$$\frac{d}{dt}\|\theta\|^2_{\mathcal{R}_\Omega L^2_X} \leq 2\epsilon_0 \|(\Pi_h^{M,n}u - u)_t\|^2_{\mathcal{R}_\Omega H^{-1}_X}$$

and

$$\|\theta(t)\|^2_{\mathcal{R}_\Omega L^2_X} \leq \|\theta(0)\|^2_{\mathcal{R}_\Omega L^2_X} + 2\epsilon_0 \int_0^t \|(\Pi_h^{M,n}u - u)_t(s)\|^2_{\mathcal{R}_\Omega H^{-1}_X} ds.$$

Due to our assumption on the initial condition, that $v_h = \Pi_h^{M,n} v$, the term $\theta(0)$ vanishes. The estimate for the second term in the last inequality is similar in some respects to the analysis for $\pi(t)$, but since the norm appears inside a time integral, it suffices to show a bound for a.e. t. Since $\Pi_h^{M,n}$ is a continuous linear map from $\mathcal{R}_\Omega H^1_{0X}$ into itself, it follows that $(\Pi_h^{M,n}u)_t = \Pi_h^{M,n}u_t$. For a.e. $s \in (0,T]$, we have that $\mathcal{A}u_t(s) + \delta_{\dot{W}}(\mathcal{M}u_t(s)) = f_t(s) - u_{tt}(s) \in \mathcal{R}'_\Omega H^{m-2}_X$. Then

$$\begin{aligned} &\|(\Pi_h^{M,n}u - u)_t(s)\|_{\mathcal{R}_\Omega H^{-1}_X} = \|\Pi_h^{M,n}u_t - u_t(s)\|_{\mathcal{R}_\Omega H^{-1}_X} \\ &\leq C_{M,n}h^{m+1}\|u_t(s)\|_{\mathcal{R}_\Omega H^m_X} + C\|f_t(s) - u_{tt}(s)\|_{\mathcal{R}'_\Omega H^{-1}_X} Q_{M,n}(\mathcal{R},\mathcal{R}') \end{aligned}$$

for a.e. s, and hence

$$\|\theta(t)\|^2_{\mathcal{R}_\Omega L^2_X} \le C^2_{M,n} h^{2(m+1)} \|u_t\|^2_{\mathcal{R}_\Omega L^2_T H^m_X} + C\|f_t - u_{tt}\|^2_{\mathcal{R}'_\Omega L^2_T H^{-1}_X} Q_{M,n}(\mathcal{R},\mathcal{R}')^2$$

for all $t \in (0, T]$.

Putting together the estimates for $\theta(t)$ and $\pi(t)$, we obtain

$$\|e_h(t)\|^2_{\mathcal{R}_\Omega L^2_X} \le C^2_{M,n} h^{2(m+1)} \left(\|u_t\|^2_{\mathcal{R}_\Omega L^2_T H^m_X} + \|u(t)\|^2_{\mathcal{R}_\Omega H^{m+2}_X}\right) + CQ_{M,n}(\mathcal{R},\mathcal{R}')^2 \left(\|f_t - u_{tt}\|^2_{\mathcal{R}'_\Omega L^2_T H^{-1}_X} + \|f(t) - u_t(t)\|^2_{\mathcal{R}'_\Omega H^{-1}_X}\right)$$

The constant C depends only on $\mathcal{R}$, $\mathcal{A}$, $\mathcal{M}$ and the elliptic estimate constant in (5.7). □

Remark 6.1. (1) If the discrete initial condition v_h is not $\Pi^{M,n}_h v$, the additional terms arising from approximating the initial error can be subsumed into the two main terms of the error estimate.

(2) If the boundary is not smooth enough, the use of regularity estimates for the stochastic adjoint problem in the proof of Proposition 5.7 will no longer hold. Thus, the application of the lower norm estimate to the term $\|(\Pi^{M,n}_h u - u)_t(s)\|_{\mathcal{R}_\Omega H^{-1}_X}$ is no longer valid. But we can nonetheless obtain a FE convergence rate of $\mathcal{O}(h^{m-1})$ in the first term of (4.5).

Appendix A. Proof of Theorem 5.6

In this section we present the proof of Theorem 5.6, which closely follows the proof in [14]. We decompose the approximation error into two components

$$\|U - U^{M,n}_h\|^2_{\mathcal{R}_\Omega H^1_{0X}} = \sum_{\alpha\in\mathcal{J}_{M,p}} \|U_\alpha - \hat{U}_\alpha\|^2_{H^1_X} r^2_\alpha + \sum_{\alpha\in\mathcal{J}\setminus\mathcal{J}_{M,p}} \|U_\alpha\|^2_{H^1_X} r^2_\alpha =: I_1 + I_2$$

For Term I_1, we follow identical steps in the proof present in the Online Supplementary Material of [14], noting that we are assuming complete knowledge of the forcing term F, to obtain

$$\|U_\alpha - \hat{U}_\alpha\|_{H^1_X} \le \hat{C}_A \inf_{v_h\in S_h} \|U_\alpha - v_h\|_{H^1_X} + \sum_{k=1}^{M} \sqrt{\alpha_k} \frac{\lambda_k}{C^{coerc}_A} \|U_{\alpha-\varepsilon_k} - \hat{U}_{\alpha-\varepsilon_k}\|_{H^1_X} \tag{A.1}$$

where $\hat{C}_A = (1 + C^b_A/C^{coerc}_A)$ and $C_k := \lambda_k/C^{coerc}_A$. Then by induction,

$$\|U_\alpha - \hat{U}_\alpha\|_{H^1_X} \le \hat{C}_A \sum_{\beta\le\alpha} c_{\alpha,\beta} \inf_{v_h\in S_h} \|U_\beta - v_h\|_{H^1_X}$$

where $c_{\alpha,\beta}$ are constants depending on α, β. The dependence is given by the following Lemma.

Lemma A.1.

$$c_{\alpha,\beta} = \frac{|\alpha-\beta|!}{\sqrt{(\alpha-\beta)!}}\sqrt{\binom{\alpha}{\beta}}\,\vec{C}^{\alpha-\beta}$$

where $\vec{C} = (C_1, C_2, \dots)$.

We will prove this later. Assuming it is true, and using (4.1), we obtain

$$\begin{aligned}
\|U_\alpha - \hat{U}_\alpha\|^2_{H^1_X} r_\alpha^2 &\le h^{2m} C_{FE}^2 \hat{C}_A^2 \left(\sum_{\beta\le\alpha} c_{\alpha,\beta}\|U_\beta\|_{H^{m+1}} r_\alpha\right)^2 \\
&\le h^{2m} C_{FE}^2 \hat{C}_A^2 \left(\sum_{\beta\le\alpha} \frac{r_\alpha^2}{r_\beta^2} c_{\alpha,\beta}^2\right)\left(\sum_{\beta\le\alpha} r_\beta^2 \|U_\beta\|^2_{H_X^{m+1}}\right) \\
&\le h^{2m} C_{FE}^2 \hat{C}_A^2 \left(\sum_{\beta\le\alpha} \binom{|\alpha|}{|\beta|}^{-1} r_{\alpha-\beta}^2 c_{\alpha,\beta}^2\right)\|U\|^2_{\mathcal{R}\Omega H_X^{m+1}}
\end{aligned}$$

So

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{J}_{M,p}} \|U_\alpha - \hat{U}_\alpha\|^2_{H^1_X} r_\alpha^2 \\
&\le h^{2m} C_{FE}^2 \hat{C}_A^2 \|U\|^2_{\mathcal{R}\Omega H_X^{m+1}} \underbrace{\left(\sum_{\alpha\in\mathcal{J}_{M,p}}\sum_{\beta\le\alpha} \binom{|\alpha|}{|\beta|}^{-1} r_{\alpha-\beta}^2 c_{\alpha,\beta}^2\right)}_{(*)}
\end{aligned}$$

To estimate $(*)$, since $\binom{|\alpha|}{|\beta|}^{-1}\binom{\alpha}{\beta} < 1$ due to Lemma B.2,

$$\begin{aligned}
(*) &= \sum_{\alpha\in\mathcal{J}_{M,p}}\sum_{\beta\le\alpha} \binom{|\alpha|}{|\beta|}^{-1} r_{\alpha-\beta}^2 \frac{|\alpha-\beta|!^2}{(\alpha-\beta)!}\binom{\alpha}{\beta}\vec{C}^{2(\alpha-\beta)} \\
&\le \sum_{\alpha\in\mathcal{J}_{M,p}}\sum_{\beta\le\alpha} (q^2\vec{C}^2)^{\alpha-\beta}\frac{|\alpha-\beta|!}{(\alpha-\beta)!} \\
&= \sum_{\beta\in\mathcal{J}_{M,p}}\sum_{\substack{\alpha\ge\beta\\ \alpha\in\mathcal{J}_{M,p}}} (q^2\vec{C}^2)^{\beta}\frac{|\beta|!}{\beta!} \\
&= \sum_{\beta\in\mathcal{J}_{M,p}} (q^2\vec{C}^2)^{\beta}\frac{|\beta|!}{\beta!} \times (\#\{\alpha\in\mathcal{J}_{M,p} : \alpha\ge\beta\}) \\
&= \sum_{n=0}^{p}\sum_{\substack{|\beta|=n\\ \dim\beta\le M}} (q^2\vec{C}^2)^{\beta}\frac{n!}{\beta!} \times \left(\binom{M+p}{M} - \frac{2^n}{\beta!}\right) \\
&\le \binom{M+p}{M}\sum_{n=0}^{p} \overline{[q]}^n_{\le M} \le \binom{M+p}{M}\frac{1}{1-\hat{q}}
\end{aligned}$$

where $\overline{[q]}_{\leq M} := \sum_{k=1}^{M} q_k^2 C_k^2 = \hat{q} - \hat{q}_W$. This gives the first term in the RHS of (5.7).

For term I_2, we use the following estimate for the H_X^1 norm of U_α, which easily follows by induction.

Lemma A.2.

$$\|U_\alpha\|_{H_X^1} \leq C_A^{ellip} \sqrt{|\alpha|!} \sum_{\beta \leq \alpha} \|F_{\alpha-\beta}\|_{H_X^{-1}} \vec{C}^\beta \sqrt{\frac{|\beta|!}{\beta!(\alpha-\beta)!}}.$$

In the rest of this section, we will write C_A in place of C_A^{ellip}. We decompose the sum in Term I_2 into

$$\sum_{\alpha \in \mathcal{J} \setminus \mathcal{J}_{M,p}} = \sum_{n=0}^{p} \sum_{i=0}^{n-1} \sum_{\left\{\alpha: \substack{|\alpha^{(1)}|=i \\ |\alpha^{(2)}|=n-i}\right\}} + \sum_{n=p+1}^{\infty} \sum_{i=0}^{n} \sum_{\left\{\alpha: \substack{|\alpha^{(1)}|=i \\ |\alpha^{(2)}|=n-i}\right\}}.$$

Consider the innermost sum

$$\sum_{\substack{|\alpha^{(1)}|=i \\ |\alpha^{(2)}|=n-i}} \|U_\alpha\|_{H_X^1}^2 r_\alpha^2 \leq \sum_{\substack{|\alpha^{(1)}|=i \\ |\alpha^{(2)}|=n-i}} C_A^2 q^\alpha \left(\sum_{\beta \leq \alpha} \|F_{\alpha-\beta}\|_{H_X^{-1}} \vec{C}^\beta \sqrt{\frac{|\beta|!}{\beta!(\alpha-\beta)!}} \right)^2$$

$$\leq C_A^2 \sum_{\substack{|\alpha^{(1)}|=i \\ |\alpha^{(2)}|=n-i}} q^\alpha \left(\sum_{\beta \leq \alpha} \bar{r}_{\alpha-\beta}^2 \|F_{\alpha-\beta}\|_{H_X^{-1}}^2 \right) \left(\sum_{\beta \leq \alpha} \bar{r}_{\alpha-\beta}^{-2} \vec{C}^{2\beta} \frac{|\beta|!}{\beta!(\alpha-\beta)!} \right)$$

$$\leq C_A^2 \|F\|_{\bar{\mathcal{R}}_\Omega H_X^{-1}}^2 \sum_{\substack{|\alpha^{(1)}|=i \\ |\alpha^{(2)}|=n-i}} \sum_{\beta \leq \alpha} (q\vec{C}^2)^\beta \left(\frac{q}{\bar{\rho}} \right)^{\alpha-\beta} \frac{|\beta|!|\alpha-\beta|!}{\beta!(\alpha-\beta)!}$$

$$= C_A^2 \|F\|_{\bar{\mathcal{R}}_\Omega H_X^{-1}}^2 \sum_{k=0}^{i} \sum_{l=0}^{n-i} \sum_{\substack{|\beta^{(1)}|=k \\ |\beta^{(2)}|=l}} \sum_{\substack{|\gamma^{(1)}|=i-k \\ |\gamma^{(2)}|=n-i-l}} (q\vec{C}^2)^\beta \left(\frac{q}{\bar{\rho}} \right)^{\alpha-\beta} \frac{|\beta|!|\alpha-\beta|!}{\beta!(\alpha-\beta)!}$$

We introduce the notation, for $\rho = (\rho_1, \rho_2, \dots)$,

$$\overline{[\rho]}_{\leq M} = \sum_{k=1}^{M} \rho_k, \qquad \overline{[\rho]}_{>M} = \sum_{k=M+1}^{\infty} \rho_k.$$

Then

$$\sum_{\substack{|\alpha^{(1)}|=i \\ |\alpha^{(2)}|=n-i}} \|U_\alpha\|_{H_X^1}^2 r_\alpha^2$$

$$\leq C_A^2 \|F\|_{\bar{\mathcal{R}}_\Omega H_X^{-1}}^2 \sum_{k=0}^{i} \sum_{l=0}^{n-i} \overline{[q\vec{C}^2]}_{\leq M}^{k} \overline{[q\vec{C}^2]}_{>M}^{l} \binom{k+l}{k} \overline{\left[\frac{q}{\bar{\rho}}\right]}_{\leq M}^{i-k} \overline{\left[\frac{q}{\bar{\rho}}\right]}_{>M}^{n-i-l} \binom{n-k-l}{i-k}$$

$$\leq C_A^2 \|F\|_{\bar{\mathcal{R}}_\Omega H_X^{-1}}^2 \binom{n}{i} \left(\overline{[q\vec{C}^2]}_{\leq M} + \overline{\left[\frac{q}{\bar{\rho}}\right]}_{\leq M} \right)^i \left(\overline{[q\vec{C}^2]}_{>M} + \overline{\left[\frac{q}{\bar{\rho}}\right]}_{>M} \right)^{n-i}$$

The rest of the proof proceeds identically to the proof in [14], and we obtain the second term in the RHS of (5.7). □

Proof of Lemma A.1. This is done by induction. Suppose

$$\|U_\gamma - \hat{U}_\gamma\|_{H^1_X} \leq \hat{C}_A \sum_{\beta \leq \gamma} c_{\gamma,\beta} \inf_{v_h \in S_h} \|U_\beta - v_h\|_{H^1_X}$$

for all $|\gamma| \leq n-1$, $\dim \gamma \leq M$. Let $|\alpha| = n$. Then the second term on the RHS of (A.1) is

$$\begin{aligned}
&\sum_{k=1}^{M} \sqrt{\alpha_k} C_k \|U_{\alpha-\varepsilon_k} - \hat{U}_{\alpha-\varepsilon_k}\|_{H^1_X} \\
&= \hat{C}_A \sum_{k=1}^{M} \sqrt{\alpha_k} C_k \sum_{\beta \leq \alpha - \varepsilon_k} c_{\alpha-\varepsilon_k,\beta} \inf_{v_h \in S_h} \|U_\beta - v_h\|_{H^1_X} \\
&= \hat{C}_A \sum_{k=1}^{M} \sqrt{\alpha_k} \sum_{\beta \leq \alpha - \varepsilon_k} \frac{|\alpha - 1 - \beta|!}{\sqrt{(\alpha - \varepsilon_k - \beta)!}} \sqrt{\binom{\alpha - \varepsilon_k}{\beta}} \vec{C}^{\alpha-\beta} \inf_{v_h \in S_h} \|U_\beta - v_h\|_{H^1_X} \\
&= \hat{C}_A \sum_{\substack{k=1 \\ \alpha_k \neq 0}}^{M} \sum_{\beta \leq \alpha - \varepsilon_k} \frac{|\alpha - 1 - \beta|!}{\sqrt{(\alpha - \beta)!}} \sqrt{\binom{\alpha}{\beta}(\alpha_k - \beta_k)} \vec{C}^{\alpha-\beta} \inf_{v_h \in S_h} \|U_\beta - v_h\|_{H^1_X} \\
&\leq \hat{C}_A \sum_{\substack{k=1 \\ \alpha_k \neq 0}}^{M} \sum_{\beta < \alpha} \frac{|\alpha - 1 - \beta|!}{\sqrt{(\alpha - \beta)!}} \sqrt{\binom{\alpha}{\beta}(\alpha_k - \beta_k)} \vec{C}^{\alpha-\beta} \inf_{v_h \in S_h} \|U_\beta - v_h\|_{H^1_X} \\
&= \hat{C}_A \sum_{\beta < \alpha} \sum_{\substack{k=1 \\ \alpha_k \neq 0}}^{M} (\alpha_k - \beta_k) \frac{|\alpha - 1 - \beta|!}{\sqrt{(\alpha - \beta)!}} \sqrt{\binom{\alpha}{\beta}} \vec{C}^{\alpha-\beta} \inf_{v_h \in S_h} \|U_\beta - v_h\|_{H^1_X} \\
&= \hat{C}_A \sum_{\beta < \alpha} \frac{|\alpha - \beta|!}{\sqrt{(\alpha - \beta)!}} \sqrt{\binom{\alpha}{\beta}} \vec{C}^{\alpha-\beta} \inf_{v_h \in S_h} \|U_\beta - v_h\|_{H^1_X} \\
&= \hat{C}_A \sum_{\beta < \alpha} c_{\alpha,\beta} \inf_{v_h \in S_h} \|U_\beta - v_h\|_{H^1_X}
\end{aligned}$$

Hence,

$$\begin{aligned}
\|U_\alpha - \hat{U}_\alpha\|_{H^1_X} &\leq \hat{C}_A \inf_{v_h \in S_h} \|U_\alpha - v_h\|_{H^1_X} + \sum_{k=1}^{M} \sqrt{\alpha_k} \frac{\lambda_k}{C_A^{coerc}} \|U_{\alpha-\varepsilon_k} - \hat{U}_{\alpha-\varepsilon_k}\|_{H^1_X} \\
&\leq \hat{C}_A \sum_{\beta \leq \alpha} c_{\alpha,\beta} \inf_{v_h \in S_h} \|U_\beta - v_h\|_{H^1_X}.
\end{aligned}$$

□

Appendix B. Some Combinatorial Results

We had used a result for the multinomial sum in infinite dimensions.

Lemma B.1. *Suppose* $\vec{\rho} = (\rho_1, \rho_2, \dots)$ *with* $\rho_k > 0$, *and let* $\overline{[\rho]} = \sum_{k\geq 1} \rho_k$. *Then for any* $n \in \mathbb{N}_0$,

$$\sum_{|\alpha|=n} \frac{\rho^\alpha}{\alpha!} = \frac{\overline{[\rho]}^n}{n!}.$$

Proof. We identify α with its characteristic set $K_\alpha = (k_1, \dots, k_{|\alpha|})$. For fixed n,

$$\begin{aligned}\sum_{|\alpha|=n} \frac{\rho^\alpha}{\alpha!} &= \sum_{k_1 \leq \dots \leq k_n} \frac{\prod_{j=1}^n \rho_{k_j}}{\alpha!} \cdot \frac{(n!/\alpha!)}{(n!/\alpha!)} \\ &= \sum_{k_1,\dots,k_n} \frac{\prod_{j=1}^n \rho_{k_j}}{\alpha!} \cdot \frac{1}{(n!/\alpha!)} \\ &= \frac{1}{n!} \sum_{k_1,\dots,k_n} \prod_{j=1}^n \rho_{k_j} = \frac{1}{n!} \Big(\sum_k \rho_k \Big)^n\end{aligned}$$

where we have multiplied by 1 and rearranged the sum over non-decreasing indices into a sum over all unordered indices. The last equality follows from the formula for the multinomial expansion. □

We had also used the combinatorial fact

Lemma B.2.

$$\frac{|\beta|!}{\beta!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} \leq \frac{|\alpha|!}{\alpha!}$$

Proof. Let $K_\alpha = (k_1, \dots, k_{|\alpha|})$ be the characteristic set of α. On the RHS, $\frac{|\alpha|!}{\alpha!}$ is the number of distinct permutations of K_α. On the LHS, we partition K_α into the two subsets corresponding to K_β and $K_{(\alpha-\beta)}$. Then, the number of distinct permutations of K_β times that of $K_{(\alpha-\beta)}$ cannot exceed the number of distinct permutations of K_α. □

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